A DCA for MPCCs converging to a S-stationary point

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Abstract: In this article we propose an Algorithm based on a DC decomposition (Difference of Convex functions) which solves MPCCs and we prove that it converges to a strongly stationary point under MPCC-LICQ.

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1 Introduction

The term MPCC stands for Mathematical Programming with Complementarity Constraints. The MPCCs are often used in economy theory. For example a competitive market with one leader, like a liberalized electricity market, can be written as a bilevel program, and therefore as a MPCC. The authors Hobbs and Pang [9] study an electricity market without thermal losses on the transmission lines, and obtain an MPCC with linear complementarity constaints. Other references about the electricity market and its interactions with the MPCCs are given [2, 4, 11, 15, 24].

One of the motivations of the MPCC is its link to bi-level programming. Consider the following problem

$$\begin{array}{ll} \min & g(x,y) \\ \text{such that} & x \in w \\ & y \in S(x), \end{array}$$

with,

$$S(x) := \arg\min_{C(x)} \varphi(x,\cdot) \text{ and } C(x) = \{y \mid G_e(x,y) = 0 \ , \ G(x,y) \le 0\}.$$

That is a bi-level program. Writing the KKT system associated with the problem $\min\limits_{C(x)} \varphi(x,\cdot)$, we obtain the following MPCC:

$$\begin{array}{ll} \min & g(x,y) \\ \text{such that} & x \in w \\ & \nabla_y \mathcal{L}(x,y,\lambda,\mu) = 0 \\ & G_e(x,y) = 0 \\ & 0 \le G(x,y) \perp \lambda \ge 0, \end{array}$$

with $\mathcal{L}$ the lagrangian function associated with the problem $\min\limits_{C(x)} \varphi(x,\cdot)$. The bi-level program and the MPCC associated are equivalent for global solutions if a Slater's constraint qualification is satisfied for the second level problem. For local solutions, the equivalence can be ensured under Slater's constraint qualification and constant rank constraint qualification [5, Dempe, Dutta, 2012].

One of the methods to solve a MPCC problem is the SQP method, which was studied for example in [7]. This method permits a numerical resolution of a problem with linear complementarity constraints with a quadratic rate of convergence under a good assumption. The authors [19] proposed a penalization method in the case where the complementarity contraints are not linear, in order to replace the nonlinear complementarity contraints by linear contraints and apply the SQP method. In [10, 12], the authors have studied a penalty method, but it only allows for a convergence to a C-stationary point under MPCC-LICQ. In a very recent work [14], the authors propose a partial penalty method and obtain which allows for a convergence to a M-stationary point under MPCC-NNAMCQ. The authors [18, 20] consider an interior point method, and obtain a super linear rate of convergence. A method of relaxation of the complementarity constraint was considered in [29], and the paper studies the stationarity properties of the limit when the relaxation parameter tends to zero. The authors [25] have worked on the relaxation and penalization of the complementarity contraint of the MPCC. They have examined the properties of distance between the solutions of the MPCC and of the relaxed problem; moreover, they have studied the boundness of the Lagrange multipliers under MPCC-LICQ. In [1] the authors prove that a second-order aumented Lagrangian method converges to a strongly stationary point if the Lagrange multipliers are bounded under MPCC-RCPLD; and converges to a C-stationary point if the Lagrange multipliers are not bounded.

A DC (Difference of Convex functions) reformulation for MPCC has been studied in many articles, for example in [13, 16, 21]. The introduction and study of the DC algorithm have been made by [30, 31, 32, 33]. In a recent work [13], the authors used the DC algorithm in LPCCs, and obtained a convergence to a weak stationary point. But at the same time, they proposed an improvement in order to avoid the weak stationary points which are not local minimizers, which helps to construct a DC algorithm which converges, under MPCC-LICQ, to a strong stationary point for MPCC. This paper aims to use a DC reformulation of a MPCC in order to obtain a new necessary and sufficient condition for a feasible point for a MPCC to be strongly stationary under MPCC-LICQ. After, the article proposes an algorithm which, under some assumptions, converges to a point which is a strongly stationary point under MPCC-LICQ if it is feasible for MPCC.

The article is organized as follows: Section 2 gives the generalities about MPCCs and the different notions of stationarity for MPCCs. Section 3 gives an optimality condition of DC programs (see e.g. [30, 31, 32, 33]). Section 4 reformulates MPCCs into a DC program. Section 5 gives an equivalent reformulation for strongly stationarity in MPCCs using the DC optimality conditions, and use it in Section 6 in order to prove the convergence of the proposed algorithm to a strong stationary point for MPCC under MPCC-LICQ.

2 Definitions and preliminary results

For this article, we work on the vectorial space $\mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^m$, with $p \in \mathbb{N}$, $m \in \mathbb{N}^*$. When $w \in \mathbb{R}^s \times \mathbb{R}^m \times \mathbb{R}^m$, we use the notation $w := (x, y, z)$, with $x \in \mathbb{R}^s$, $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^m$.

We consider the following MPCC:

$$\begin{array}{ll} \min & f(w) \\ \text{subject to} & g(w) \leq 0 \ , \ h(w) = 0 \\ & 0 \leq y \perp z \geq 0 \end{array} \tag{2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^q$, $h : \mathbb{R}^n \to \mathbb{R}^r$ with $n := p + 2m$.

The functions g_i are supposed C^1 and convex on their domain, and h is supposed to be affine. Therefore, the set

$$\Omega := \{w =: (x, y, z) \in \mathbb{R}^n \mid g(w) \leq 0 \ , \ h(w) = 0 \ , \ y \geq 0 \ , \ z \geq 0\} \tag{2.2}$$

is a convex set. The MPCC can be written as

$$\begin{array}{ll} \min & f(w) \\ \text{subject to} & w \in \Omega \cap \Delta \end{array}$$

where

$$\Delta := \{w := (x, y, z) \in \mathbb{R}^n \mid \langle y, z \rangle = 0\} . \tag{2.3}$$

The function f is supposed to be a C^1 and DC function, which means that there exist two C^1 and convex functions $f_1 : \Omega \to \mathbb{R}$ and $f_2 : \Omega \to \mathbb{R}$ be such that:

$$f := f_1 - f_2. \tag{2.4}$$

The following definition allows for the normal cone to Ω to be written with Lagrange multipliers.

Definition 2.1. We say that the constraint set Ω is **qualified** at a point $\overline{w} \in \Omega$ if the following inclusion holds:

$$N_\Omega(\overline{w}) \subset \left\{ \nabla g(\overline{w})^T \lambda^g + \nabla h^T(\overline{w})\lambda^h - \begin{pmatrix} 0 \\ \nu_1 \\ \nu_2 \end{pmatrix} \;\middle|\; \begin{array}{l} \lambda^h \in \mathbb{R}^q \ , \ 0 \geq g(\overline{w}) \perp \lambda^g \geq 0 \\ 0 \leq \nu_1 \perp \overline{y} \geq 0 \ , \ 0 \leq \nu_2 \perp \overline{z} \geq 0. \end{array} \right\}$$

For example if LICQ or MFCQ holds at $\overline{w} \in \Omega$, or if the perturbated set-valued map

$$M(y) = \{w \in \mathbb{R}^n \mid G(w) + y \in D\},$$

is calm at $(0, \overline{w})$, with $G(w) := (g(w), h(w), y, z)$ and $D := (\mathbb{R}_-)^p \times \{0\}^q \times (\mathbb{R}_+)^{2m}$, then Ω is qualified at $\overline{w} \in \Omega$.

Given $\overline{w} \in \Omega$, we define the following index sets of active and inactive constraints:

$$\begin{array}{rcl} I_g(\overline{w}) & := & \{i \in \{1, \cdots, p\} \mid g_i(\overline{w}) = 0\} \\ I_g^c(\overline{w}) & := & \{i \in \{1, \cdots, p\} \mid g_i(\overline{w}) < 0\} \\ I_y(\overline{w}) & := & \{i \in \{1, \cdots, m\} \mid \overline{y}_i = 0\} \\ I_z(\overline{w}) & := & \{i \in \{1, \cdots, m\} \mid \overline{z}_i = 0\} \end{array} \tag{2.5}$$

Observe that $\overline{w} \in \Omega$ is feasible for MPCC (2.1) if and only if $I_y(\overline{w}) \cup I_z(\overline{w}) = \{1, \cdots, m\}$. For this class of problem, there exist many notions of stationary points; two are: the weakly and the strongly, which are defined below.

Definition 2.2. A feasible point $\overline{w}$ of the MPCC is said to be **weakly stationary** if there exists a vector of MPCC multipliers $(\lambda_g, \lambda_h, \hat{\nu}_1, \hat{\nu}_2)$ such that:

$$\begin{array}{rcl} \nabla f(\overline{w}) + \nabla g(\overline{w})^T \overline{\lambda}^g + \nabla h(\overline{w})^T \overline{\lambda}^h - (0, \hat{\nu}_1, \hat{\nu}_2) & = & 0 \\ h(\overline{w}) = 0,\ g(\overline{w}) \leq 0,\ \overline{\lambda}^g \geq 0,\ \langle \overline{\lambda}^g, g(\overline{w}) \rangle & = & 0 \\ \forall i \notin I_y(\overline{w}) :\ \hat{\nu}_{1,i} & = & 0 \\ \forall i \notin I_z(\overline{w}) :\ \hat{\nu}_{2,i} & = & 0. \end{array}$$

In addition, the feasible vector $\overline{w}$ is called a **strongly stationary** point if $\hat{\nu}_{1,i} \geq 0,\ \hat{\nu}_{2,i} \geq 0$, for all $i \in I_y(\overline{w}) \cap I_z(\overline{w})$.

Associated with any given feasible vector $\overline{w}$ of MPCC (2.1), there is a nonlinear program called the *tightened* NLP (TNLP($\overline{w}$)):

$$\begin{array}{ll} \min & f(w) \\ \text{subject to} & g(w) \leq 0\ ,\ h(w) = 0 \\ & y_i = 0,\ \forall i \in I_y(\overline{w}) \\ & y_i \geq 0,\ \forall i \notin I_y(\overline{w}) \\ & z_i = 0,\ \forall i \in I_z(\overline{w}) \\ & z_i \geq 0,\ \forall i \notin I_z(\overline{w}) \end{array} \tag{2.6}$$

Note that a feasible point of MPCC (2.1) $\overline{w}$ is weakly stationary if and only if there exists a vector MPCC multipliers $\lambda = (\lambda^g, \lambda^h, \hat{\nu}_1, \hat{\nu}_2)$ such that $(\overline{w}, \lambda)$ is a KKT stationary point of the TNLP (2.6). The following definition gives a very important constraint qualification for MPCC.

Definition 2.3. We say that MPCC-LICQ holds at a feasible point $\overline{w}$ if for any $(\lambda^g, \lambda^h, \nu_1, \nu_2) \in \mathbb{R}^{q+r+2m}$, the following implication is true:

$$\left.\begin{array}{rcl} \nabla g(\overline{w})^T \lambda^g + \nabla h(\overline{w})^T \lambda^h - (0, \nu_1, \nu_2) & = & 0 \\ \langle \lambda^g, g(\overline{w}) \rangle & = & 0 \\ \forall i \notin I_y(\overline{w}) :\ \nu_{1,i} & = & 0 \\ \forall i \notin I_z(\overline{w}) :\ \nu_{2,i} & = & 0. \end{array}\right\} \Longrightarrow (\lambda^g, \lambda^h, \nu_1, \nu_2) = (0, 0, 0, 0).$$

A very important link between the solutions of MPCC (2.1) and the strongly stationary points:

Theorem 2.4. *[23] If the MPCC-LICQ holds at a local minimizer $\overline{w}$ of the MPCC, then $\overline{w}$ is a strongly stationary point of MPCC.*

We now define the B(ouligand)-stationarity for MPCCs.

Definition 2.5. A feasible point $\overline{w}$ is said to be a B-stationary point if 0 solves the following *Linear Program with Complementarity Constraints*, with the vector $d \in \mathbb{R}^n$ being the decision variable:

$$\begin{array}{ll} \min & \langle \nabla f(\overline{w}), d \rangle \\ \text{subject to} & g(\overline{w}) + \nabla g(\overline{w}) d \leq 0\ ,\ h(\overline{w}) + \nabla h(\overline{w}) d = 0 \\ & 0 \leq \overline{y} + d_1 \perp \overline{z} + d_2 \geq 0 \end{array} \tag{2.7}$$

The B-stationarity for MPCCs are related to the strongly stationary points:

Theorem 2.6. *[28] If a feasible point for the MPCC is a strong stationary point of the MPCC, then it is a B-stationary point. Conversely, if $\overline{w}$ is a B-stationary point of the MPCC, and MPCC-LICQ holds at $\overline{w}$, then $\overline{w}$ is a strongly stationary point of MPCC.*

3 Generalities about the minimization of the difference of convex functions

In this section we give some results about the minimization of the difference of convex functions. Let X be a Banach space, $u : X \to \mathbb{R} \cup \{+\infty\}$ and $v : X \to \mathbb{R} \cup \{+\infty\}$ two convex, proper and continous functions. We consider the following problem

$$\min_{x \in X} u(x) - v(x) \tag{3.1}$$

which is a DC (Difference of Convex functions) problem. By convention $+\infty - (+\infty) = +\infty$.

If $\psi : \mathbb{R}^n \to \mathbb{R}$ is a proper function, then we define its subdifferential at $\overline{x} \in \mathrm{dom}(\psi)$ as follows:

$$\partial\psi(\overline{x}) := \{x^* \in \mathbb{R}^n \mid \langle x^*, x - \overline{x}\rangle \leq f(x) - f(\overline{x})\}.$$

If ψ is proper, continous and convex on its domain, then for every $\overline{x} \in \mathrm{dom}(\psi)$, $\partial\psi(\overline{x}) \neq \emptyset$.

The classical first order optimality condition is $0 \in \partial(u - v)(\overline{x})$, which leads to $\partial u(\overline{x}) \cap \partial v(\overline{x}) \neq \emptyset$. In DC program, there is a stronger first order optimality condition, which is given in the follow proposition. For more informations, see e.g. [31, 33].

Proposition 3.1. *We suppose that dom$(u) \subset$ dom(v). A necessary condition for $\overline{x}$ to be a solution of problem (3.1) is*

$$\partial v(\overline{x}) \subset \partial u(\overline{x}).$$

Moreover, if v is a polyhedral convex function, the above inclusion is a sufficient condition for $\overline{x}$ to be a solution of (3.1).

We recall that a function $g : X \to \mathbb{R} \cup \{+\infty\}$ is a polyhedral convex function if there exist an integer $p \in \mathbb{N}$, elements $a_1, \cdots, a_p$ of X^*, some reals $b_1, \cdots, b_p$, a polyhedral convex set $S \subset X$ such that

$$\forall x \in X \ , \ g(x) = \max_{i=1,\cdots,p} (\langle a_i, x\rangle + b_i) + \delta_S(x)$$

where $\delta_S(x) = \begin{cases} 0 & \text{if} \quad x \in S \\ +\infty & \text{if} \quad x \notin S \end{cases}$.

The assumption $\mathrm{dom}(u) \subset \mathrm{dom}(v)$ ensure that for every $x \in X$, we have $u(x) - v(x) > -\infty$.

We can observe that the inclusion $\partial v(\overline{x}) \subset \partial u(\overline{x})$ is stronger than the classical first order optimality condition $0 \in \partial(u - v)(\overline{x})$. In Section 5, we will relate in MPCC the $\partial v(\overline{x}) \subset \partial u(\overline{x})$ with the strong stationarity and $0 \in \partial(u - v)(\overline{x})$ with the weak stationarity.

4 Reformulation of the MPCC into DC program

We come back to MPCC problem (2.1). The main idea is to penalize the constraint of complementarity and to formulate the MPCC into a DC program.

We recall that MPCC (2.1) has the following form

$$\begin{array}{ll} \min & f(w) \\ \text{subject to} & x \in \Omega \cap \Delta \end{array} \tag{4.1}$$

where Ω and Δ given by (2.2) and (2.3). We recall that Ω is a convex set because g_i are convex functions and h is an affine function. Moreover, Ω and Δ are closed sets.

We define the constant $\alpha \geq 0$ by:

$$\alpha = \inf\left\{\frac{\mathrm{dist}(w, \Delta)}{\mathrm{dist}(w, \Omega \cap \Delta)} \ \mid w \in \Omega \setminus \Delta\right\}. \tag{4.2}$$

We can easily verify that $\alpha \in [0, 1]$.

In what follows we consider a family of functions $\Phi : \mathbb{R}^n \to \mathbb{R}$ satisfying the following hypothesis:

$(\mathcal{H}_1)$ For all $w := (x, y, z) \in \Omega$, $\Phi(w) = 0 \Longleftrightarrow \langle y, z\rangle = 0$. Moreover, $\Phi(\cdot, y, z)$ is a constant function on its domain.

($\mathcal{H}_2$) There exists a constant $c > 0$ such that for all $w \in \Omega$, one has $\Phi(w) \geq c\mathrm{dist}(w, \Delta)$, where Δ is given in (2.3).

($\mathcal{H}_3$) The function Φ is concave on the set $\{w := (x, y, z) \in \mathbb{R}^n \mid y \geq 0 \text{ and } z \geq 0\}$.

($\mathcal{H}_4$) The subdifferential $\partial(-\Phi)$ is uniformly bounded on the set $\{w := (x, y, z) \in \mathbb{R}^n \mid y \geq 0 \text{ and } z \geq 0\}$.

Example 4.0.1. The functions

$$\Phi(x, y, z) = \sum_{i=1}^{m} \min\{y_i, z_i\} \tag{4.3}$$

and

$$\Psi(x, y, z) = \sum_{i=1}^{m} \left(y_i + z_i - \sqrt{y_i^2 + z_i^2} \right) \tag{4.4}$$

satisfy all the above hypotheses.

The following proposition allows for a partial penalization of (4.1).

Proposition 4.1. *Suppose that $\alpha > 0$, where α is given in (4.2). Suppose that $f : \Omega \to \mathbb{R}$ is Lipschitz continous with a constant of Lipschitz $L \geq 0$. We consider a function Φ which satisfies the hypothesis $\mathcal{H}_1, \cdots, \mathcal{H}_4$. Let $\mu > \frac{L}{c\alpha}$, with $c > 0$ the constant of the hypothesis $\mathcal{H}_2$. The optimization problem*

$$\begin{array}{ll} \min & f(w) + \mu\Phi(w) \\ \textit{subject to} & w \in \Omega \end{array} \tag{4.5}$$

admits the same solutions as MPCC (2.1).

Proof. Suppose that $\overline{w}$ is a solution of MPCC, which implies that $\overline{w}$ is a solution of (4.1) with Ω and Δ given by (2.2) and (2.3). Let $w \in \Omega$ and $u \in \Omega \cap \Delta$ be such that $\|w - u\| = \mathrm{dist}(x, \Omega \cap \Delta)$. Since $f(u) \geq f(\overline{w})$ and $\mu > L/c\alpha$, we have:

$$\begin{array}{rcl} f(w) + \mu\Phi(w) & \geq & f(w) + \mu c\mathrm{dist}(w, \Delta) \text{ by } \mathcal{H}_2 \\ & \geq & f(w) + \mu c\alpha\mathrm{dist}(w, \Omega \cap \Delta) \text{ by definition of } \alpha \\ & = & f(w) + \mu c\alpha\|w - u\| \\ & \geq & f(w) + \dfrac{\mu c\alpha}{L}|f(u) - f(w)| \\ & \geq & f(w) + |f(u) - f(w)| \text{ because } \mu > \dfrac{L}{c\alpha} \\ & \geq & f(w) + f(u) - f(w) \\ & = & f(u) \\ & \geq & f(\overline{w}) \\ & = & f(\overline{w}) + \mu\mathrm{dist}(\overline{w}, \Delta). \end{array}$$

That is true for all $w \in \Omega$, then $\overline{w}$ is a solution of (4.5).

Conversely, assume that $\overline{w}$ is a solution of (4.5). We first show that $\overline{w} \in \Omega \cap \Delta$. Suppose that $\overline{w} \notin \Delta$ and let $w \in \Omega \cap \Delta$ be such that $\|w - \overline{w}\| = \mathrm{dist}(\overline{w}, \Omega \cap \Delta)$. We have:

$$\begin{array}{rcl} f(w) + \mu\Phi(w) & = & f(w) \text{ by } \mathcal{H}_1 \\ & = & f(w) - f(\overline{w}) + f(\overline{w}) \\ & \leq & L\|w - \overline{w}\| + f(\overline{w}) \\ & = & L\mathrm{dist}(\overline{w}, \Omega \cap \Delta) + f(\overline{w}) \\ & \leq & \dfrac{L}{\alpha}\mathrm{dist}(\overline{w}, \Delta) + f(\overline{w}) \text{ by definition of } \alpha \text{ given by (4.2)} \\ & \leq & \dfrac{L}{c\alpha}\Phi(\overline{w}) + f(\overline{w}) \text{ by } \mathcal{H}_2 \\ & < & \mu\Phi(\overline{w}) + f(\overline{w}). \end{array}$$

The last inequality results from the inequality $\mu > \frac{L}{c\alpha}$ and from the fact that by $\mathcal{H}_2$ we have $\Phi(\overline{w}) > 0$ because $\overline{w} \in \Omega \setminus \Delta$. We obtain a contradiction with $\overline{w} \in \arg\min_{\Omega} f + \mu\Phi$, then $\overline{w} \in \Omega \cap \Delta$.

Finally, for any $w \in \Omega \cap \Delta$, we have:

$$\begin{array}{rcl} f(w) & = & f(w) + \mu\Phi(w) \\ & \geq & f(\overline{w}) + \mu\Phi(\overline{w}) \text{ because } \overline{w} \in \arg\min\limits_{\Omega} f + \mu\Phi \\ & = & f(\overline{w}). \end{array}$$

That proves that $\overline{w}$ is a solution of (4.1) which is MPCC (2.1). □

We recall that f is a DC function, which means that $f = f_1 - f_2$, with f_1 and f_2 two convex functions. By the concavity of Φ and by Proposition 4.1, if f is Lipschitz-continuous on Ω and $\alpha > 0$, where α is given by (4.2), then for all μ large enough, MPCC problem (2.1) can be written as the following DC program:

$$\min_{w \in \mathbb{R}^n} (f_1 + \delta_\Omega)(w) - (f_2 - \mu\Phi)(w). \tag{4.6}$$

We recall that the function δ_Ω is defined as follows:

$$\forall w \in \mathbb{R}^n \ , \ \delta_\Omega(w) := \begin{cases} 0 & \text{if} \quad w \in \Omega \\ +\infty & \text{if} \quad w \notin \Omega. \end{cases}$$

We finish this section with a sufficient and necessary condition for α given in (4.2) to be not equal to zero, in the case where Ω is bounded and closed.

Proposition 4.2. *We suppose that Ω is bounded and closed. Then $\alpha = 0$ if and only if there exists an element $\overline{w} \in \Omega \cap \Delta$, a sequence $(w^k) \in (\Omega \setminus \Delta)^{\mathbb{N}}$ satisfying $\lim w^k = \overline{w}$ and*

$$\lim_{k \to +\infty} \frac{dist(w^k, \Delta)}{dist(w^k, \Omega \cap \Delta)} = 0.$$

Proof. By definition of α, if $\alpha = 0$ then there exists a sequence $(w^k) \in (\Omega \setminus \Delta)^{\mathbb{N}}$ satisfying

$$\lim_{n \to +\infty} \frac{\text{dist}(w^k, \Delta)}{\text{dist}(w^k, \Omega \cap \Delta)} = 0.$$

Since the set Ω is compact, there exists a subsequence of $(w^k)_k$ converging to an element $\overline{w} \in \Omega$.
Without losing generality, we can suppose that the whole sequence (w^k) converges. Since the sequence (w^k) is bounded, the sequence $(\text{dist}(w^k, \Delta \cap \Omega))_k$ is also bounded, thus $\lim_{k \to \infty} \text{dist}(w^k, \Delta) = 0$. Therefore, by continuity of $\text{dist}(\cdot, \Delta)$, we obtain $\text{dist}(\overline{w}, \Delta) = 0$, thus $\overline{w} \in \Delta$, and then $\overline{w} \in \Omega \cap \Delta$.

The converse is clear by the definition of α. □

From the previous proposition, we can deduce that if Ω is bounded, h and g are affine functions, then $\alpha > 0$. Before we define the calmness of a multifunction.

Definition 4.3. Let $T : X \rightrightarrows Y$ a multifunction, where X and Y are Banach spaces. Set $(\overline{x}, \overline{y}) \in \text{Gr}(T)$ (which means that $\overline{y} \in T(\overline{x})$). We say that T is calm at $(\overline{x}, \overline{y})$ if there exist $r, \varepsilon, L > 0$ be such that:

$$\forall x \in B(\overline{x}, r) \ , \ T(x) \cap B(\overline{y}, \varepsilon) \subset T(\overline{x}) + B(0, L\|x - \overline{x}\|).$$

We can now prove the following corollary.

Corollary 4.4. *Assume that Ω is bounded and closed, and h and g are affine functions. Then $\alpha > 0$.*

Proof. Let $\overline{w} \in \Omega \cap \Delta$ and a sequence $w^k \to \overline{w}$ with $w^k \in \Omega \setminus \Delta$. Let us define the function

$$G(w) := \left(h(w), g(w), y, z, \sum_{i=1}^{m} \min(y_i, z_i) \right)$$

and the multifunction

$$M(u) := \{ w \in \Omega \ : \ G(w) + u \in D \}$$

with $D := \mathbb{R}^p_- \times \{0\}^q \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \{0\}$. We observe that $M(0) = \Omega \cap \Delta$.

Since h and g are affine functions, the multifunction M is polyhedral (because its graph is the union of polyhedral sets), then it is calm at $(0, \overline{w})$ by Robinson's Theorem [26]. By definition of calmness, there exist $r, \varepsilon, L > 0$ be such that:

$$\forall u \in B(0, r) \ , \ M(u) \cap B(\overline{w}, \varepsilon) \subset M(0) + B(0, L\|u\|). \tag{4.7}$$

For each k, we consider $d^k \in D$ be such that $\|G(w^k) - d^k\| = \text{dist}(G(w^k), D)$. The function $\text{dist}(G(\cdot), D)$ is continuous, then $\text{dist}(G(w^k), D) \to \text{dist}(G(\overline{w}), D) = 0$ because $\overline{w} \in \Omega \cap \Delta$. We deduce that $\|G(w^k) - d^k\| < r$ for each k large enough. We deduce from (4.7) that:

$$M(d^k - G(w^k)) \cap B(\overline{w}, \varepsilon) \subset M(0) + B(0, L\|d^k - G(w^k)\|).$$

Given that $G(w^k) + d^k - G(w^k) = d^k \in D$, we have $w^k \in M(d^k - G(w^k))$. Since $w^k \to \overline{w}$, we deduce that $w^k \in M(d^k - G(w^k)) \cap B(\overline{w}, \varepsilon)$ for all k large enough. Then, for all k large enough, we have:

$$w^k \in M(0) + B(0, L\|d^k - G(w^k)\|).$$

Since $M(0) = \Omega \cap \Delta$, we have

$$\text{dist}(w^k, \Omega \cap \Delta) \leq L\|d^k - G(w^k)\| = L\text{dist}(G(w^k), D).$$

Given that $w^k \in \Omega$, we have $h(w^k) = 0$, $g(w^k) \leq 0$, $y^k \geq 0$, $z^k \geq 0$, we have

$$\begin{aligned}
\text{dist}(G(w^k), D) &= \sum_{i=1}^{m} \min(y_i^k, z_i^k) \\
&\leq m \left(\sum_{i=1}^{m} \min(y_i^k, z_i^k)^2 \right)^{\frac{1}{2}} \quad \text{by the Cauchy-Scharz inequality} \\
&= m\text{dist}(w^k, \Delta).
\end{aligned}$$

We finally obtain that $\text{dist}(w^k, \Omega \cap \Delta) \leq Lm\text{dist}(w^k, \Delta)$, then:

$$\frac{\text{dist}(w^k, \Delta)}{\text{dist}(w^k, \Omega \cap \Delta)} \geq \frac{1}{Lm} > 0.$$

That is true for all $\overline{w} \in \Omega \cap \Delta$ and for all $w^k \to \overline{w}$ with $w^k \in \Omega \setminus \Delta$, then by Proposition 4.2, we have $\alpha > 0$. ☐

5 A new characterization for stationarity in MPCC considering Optimality Conditions for DC programs

If $\overline{w}$ is a local solution of (2.1) and MPCC-LICQ holds at this point, then $\overline{w}$ is a strongly stationary point by Theorem 2.4, but moreover $\overline{w}$ solves (4.6), therefore according to Proposition 3.1, the point $\overline{w}$ satisfies $\partial(f_2 - \mu\Phi)(\overline{w}) \subset \partial(f_1 + \delta_\Omega)(\overline{w})$. If the function f is differentiable, by the equality $f = f_1 - f_2$, it is equivalent to

$$\partial(-\mu\Phi)(\overline{w}) \subset \nabla f(\overline{w}) + N_\Omega(\overline{w}). \tag{5.1}$$

In this section we will show that under MPCC-LICQ, $\overline{w}$ is a strongly stationary point for MPCC if and only if inclusion (5.1) is true with Φ satisfying hypotheses $\mathcal{H}_1$-$\mathcal{H}_4$ and an additional assumption. The following technical lemma is useful for the proof of Proposition 5.3 and Theorem 5.4.

Lemma 5.1. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ satisfying hypotheses $\mathcal{H}_1$-$\mathcal{H}_4$, let $c > 0$ be a constant for Φ in $\mathcal{H}_2$ and let $\overline{w}$ be a feasible point for MPCC (2.1). Assume that the set*

$$\left\{ \left((y_i^*)_{i \in I_y(\overline{w}) \setminus I_z(\overline{w})}, (z_i^*)_{i \in I_z(\overline{w}) \setminus I_y(\overline{w})} \right) \mid w^* := (x^*, y^*, z^*) \in \partial(-\Phi)(\overline{w}) \right\} \tag{5.2}$$

is a singleton. Then there exists a polyhedral concave function $\tilde{\Phi}$ which satisfies hypotheses $\mathcal{H}_1$-$\mathcal{H}_4$ with the same constant $c > 0$ in $\mathcal{H}_2$ and such that:

$$\partial(-\tilde{\Phi})(\overline{w}) \subset \partial(-\Phi)(\overline{w}).$$

Remark 5.2. The assumption (5.2) is satisfied for example by the function given in (4.4).

Proof. The proof will be divided into two steps.

Step 1: We prove that there exist two elements $w^* = (x^*, y^*, z^*) \in \partial(-\Phi)(\overline{w})$ and $\overline{w}^* = (\overline{x}^*, \overline{y}^*, \overline{z}^*) \in \partial(-\Phi)(\overline{w})$ satisfying:

$$\begin{cases}
x^* = \overline{x}^* = 0 \\
y_i^* = \overline{y}_i^* \leq -c \text{ and } z_i^* = \overline{z}_i^* = 0 \text{ if } i \in I_y(\overline{w}) \setminus I_z(\overline{w}) \\
y_i^* = \overline{y}_i^* = 0 \text{ and } z_i^* = \overline{z}_i^* \leq -c \text{ if } i \in I_z(\overline{w}) \setminus I_y(\overline{w}) \\
y_i^* \leq -c \ , \ z_i^* = 0 \ , \ \overline{y}_i^* = 0, \ \overline{z}_i^* \leq -c \text{ if } i \in I_y(\overline{w}) \cap I_z(\overline{w}),
\end{cases}$$

with c being the constant defined by $\mathcal{H}_2$.

Remember that for $w \in \mathbb{R}^n$, we write $w := (x, y, z)$, where $x \in \mathbb{R}^s$, $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^m$ with $n = s+2m$. Given that $\overline{w}$ is a feasible point for MPCC (2.1), we have $\overline{w} \in \Delta$, then hypothesis $\mathcal{H}_1$ implies that $\Phi(\overline{w}) = 0$.

We first assume that $I_y(\overline{w}) \cap I_z(\overline{w}) = \emptyset$. Let $w^* = (x^*, y^*, z^*) \in \partial(-\Phi)(\overline{w})$. Clearly, by $\mathcal{H}_1$, we have $x^* = 0$. Let $i \in I_y(\overline{w})$. We show that $y_i^* \leq -c$ and $z_i^* = 0$.

Consider the vector $w := (\overline{x}, \overline{y}, z)$ with $z := (\overline{z}_1, \cdots, \overline{z}_{i-1}, z_i, \overline{z}_{i+1}, \cdots, \overline{z}_m)$ and $z_i > 0$. Given that $\langle \overline{y}, \overline{z} \rangle = 0$ and $\overline{y}_i = 0$, we have $\langle \overline{y}, z \rangle = 0$, then by $\mathcal{H}_1$, we have $\Phi(w) = 0$. Therefore:

$$\langle w^*, w - \overline{w} \rangle \leq -\Phi(w) + \Phi(\overline{w}) = 0.$$

At the same time, we have $\langle w^*, w - \overline{w} \rangle = z_i^*(z_i - \overline{z}_i)$, which implies that $z_i^*(z_i - \overline{z}_i) \leq 0$ for all $z_i > 0$. Given that $\overline{z}_i > 0$ (because $I_y(\overline{w}) \cap I_z(\overline{w}) = \emptyset$ and $i \in I_y(\overline{w})$, then $i \notin I_z(\overline{w})$), we can chose $z_i \in]0, \overline{z}_i[$, which leads to $z_i^* \geq 0$. If we chose $z_i > \overline{z}_i$, we then obtain $z_i^* \leq 0$, which finally proves that $z_i^* = 0$.

Consider now the vector $w := (\overline{x}, y, \overline{z})$ with $y := (\overline{y}_1, \cdots, \overline{y}_{i-1}, y_i, \overline{y}_{i+1}, \cdots, \overline{y}_m)$ and $y_i > 0$. Given that $\overline{z}_i > 0$, we have $\langle y, \overline{z} \rangle > 0$, which proves that $w \notin \Delta$. We can easily see that $\mathrm{dist}(w, \Delta) = y_i$, then the hypothesis $\mathcal{H}_2$ gives $\Phi(w) \geq c y_i$. This implies that:

$$\langle w^*, w - \overline{w} \rangle \leq -\Phi(w) + \Phi(\overline{w}) \leq -c y_i.$$

At the same time, we have $\langle w^*, w - \overline{w} \rangle = y_i^*(y_i - \overline{y}_i) = y_i^* y_i$ because $\overline{y}_i = 0$, which implies that $y_i^* y_i \leq -c y_i$ for all $y_i > 0$. We finally obtain that $y_i^* \leq -c$.

In the same way, we prove that if $i \in I_z(\overline{w})$, then $y_i^* = 0$ and $z_i^* \leq -c$. Setting $\overline{w}^* := w^*$, we obtain the two elements of $\partial(-\Phi)(\overline{w})$ which satisfy

$$\begin{cases} x^* = \overline{x}^* = 0 \\ y_i^* = \overline{y}_i^* \leq -c \text{ and } z_i^* = \overline{z}_i^* = 0 \text{ if } i \in I_y(\overline{w}) \setminus I_z(\overline{w}) \\ y_i^* = \overline{y}_i^* = 0 \text{ and } z_i^* = \overline{z}_i^* \leq -c \text{ if } i \in I_z(\overline{w}) \setminus I_y(\overline{w}) \\ y_i^* \leq -c \text{ , } z_i^* = 0 \text{ , } \overline{y}_i^* = 0, \overline{z}_i^* \leq -c \text{ if } i \in I_y(\overline{w}) \cap I_z(\overline{w}). \end{cases}$$

We suppose now that $I_y(\overline{w}) \cap I_z(\overline{w}) \neq \emptyset$. Consider a sequence $w^n \to \overline{w}$ be such that for all $i \in I_y(\overline{w}) \cap I_z(\overline{w})$, we have $i \in I_y(w^n) \setminus I_z(w^n)$. Let a sequence $w^{*,n} \in \partial(-\Phi)(w^n)$. By $\mathcal{H}_4$, the sequence $(w^{*,n})_n$ is bounded, then it as a converging subsequence, which we denote by w^* its limit. Given that the subdifferential of a convex function has a closed graph, we have $w^* \in \partial(-\Phi)(\overline{w})$. For each $i \in I_y(\overline{w}) \setminus I_z(\overline{w})$, we have $i \in I_y(w^n) \setminus I_z(w^n)$ for all n large enough because $w^n \to \overline{w}$, then according to the previous case, $y_i^{*,n} \leq -c$ and $z_i^{*,n} = 0$. Passing to the limit, we have $y_i^* \leq -c$ and $z_i^* = 0$. For each $i \in I_y(\overline{w}) \cap I_z(\overline{w})$, we have $i \in I_y(w^n) \setminus I_z(w^n)$ by construction of w^n, then using the same arguments as before, we prove that $y_i^* \leq -c$ and $z_i^* = 0$. For each $i \in I_z(\overline{w}) \setminus I_y(\overline{w})$, we prove that $y_i^* = 0$ and $z_i^* \leq -c$ using the same arguments as before.

Consider now a sequence $\overline{w}^n \to \overline{w}$ be such that for all $i \in I_y(\overline{w}) \cap I_z(\overline{w})$, we have $i \in I_z(\overline{w}^n) \setminus I_y(\overline{w}^n)$. Let a sequence $\overline{w}^{*,n} = (\overline{x}^{*,n}, \overline{y}^{*,n}, \overline{z}^{*,n}) \in \partial(-\Phi)(\overline{w}^n)$. By $\mathcal{H}_4$, the sequence $(\overline{w}^{*,n})_n$ is bounded, then it as a converging subsequence, which we denote by $\overline{w}^* := (\overline{x}, \overline{y}, \overline{z})$ its limit. Given that the subdifferential of a convex function has a closed graph, we have $\overline{w}^* \in \partial(-\Phi)(\overline{w})$. For each $i \in I_z(\overline{w}) \setminus I_y(\overline{w})$, we have $i \in I_z(\overline{w}^n) \setminus I_y(\overline{w}^n)$ for all n large enough because $\overline{w}^n \to \overline{w}$, then according to the previous case, $\overline{y}_i^{*,n} = 0$ and $\overline{z}_i^{*,n} \leq -c$. Passing to the limit, we have $\overline{y}_i^* = 0$ and $\overline{z}_i^* \leq -c$. For each $i \in I_y(\overline{w}) \cap I_z(\overline{w})$, we have $i \in I_z(\overline{w}^n) \setminus I_y(\overline{w}^n)$ by construction of $\overline{w}^n$, then using the same arguments as before, we prove that $\overline{y}_i^* = 0$ and $\overline{z}_i^* \leq -c$. For each $i \in I_y(\overline{w}) \setminus I_z(\overline{w})$, we prove that $\overline{y}_i^* \leq -c$ and $\overline{z}_i^* = 0$ using the same arguments as before.

We have then constructed two elements w^* and $\overline{w}^*$ in $\partial(-\Phi)(\overline{w})$. Given that the set (5.2) is a singleton, we deduce that for each $i \in \Big(I_y(\overline{w}) \setminus I_z(\overline{w})\Big) \cup \Big(I_z(\overline{w}) \setminus I_y(\overline{w})\Big)$, we have $y_i^* = \overline{y}_i^*$ and $z_i^* = \overline{z}_i^*$. Finally w^* and $\overline{w}^*$ satisfy:

$$\begin{cases} x^* = \overline{x}^* = 0 \\ y_i^* = \overline{y}_i^* \leq -c \text{ and } z_i^* = \overline{z}_i^* = 0 \text{ if } i \in I_y(\overline{w}) \setminus I_z(\overline{w}) \\ y_i^* = \overline{y}_i^* = 0 \text{ and } z_i^* = \overline{z}_i^* \leq -c \text{ if } i \in I_z(\overline{w}) \setminus I_y(\overline{w}) \\ y_i^* \leq -c \text{ , } z_i^* = 0 \text{ , } \overline{y}_i^* = 0, \overline{z}_i^* \leq -c \text{ if } i \in I_y(\overline{w}) \cap I_z(\overline{w}), \end{cases}$$

Step 2: We now construct the function $\tilde{\Phi}$. Let the function $\tilde{\Phi}$ be defined as follows:

$$
\begin{aligned}
\tilde{\Phi}(w) \quad := \quad & \sum_{i\in I_y(\overline{w})\setminus I_z(\overline{w})} -y_i^* \min(y_i,z_i) \\
+ \quad & \sum_{i\in I_z(\overline{w})\setminus I_y(\overline{w})} -z_i^* \min(y_i,z_i) \\
+ \quad & \sum_{i\in I_y(\overline{w})\cap I_z(\overline{w})} \min(-y_i^* y_i, -\overline{z}_i^* z_i).
\end{aligned}
$$

We can easily verify that for all $x\in\mathbb{R}^n$, if $y\geq 0$ and $z\geq 0$, then $\tilde{\Phi}(w)\geq c\sum_{i=1}^m \min(y_i,z_i)\geq 0$, and $\tilde{\Phi}(w)=0$ if and only if $\langle y,z\rangle=0$. This ensures that hypothesis $\mathcal{H}_1$ is satisfied.

Since for all nonegative reals $\alpha_1,\cdots,\alpha_m$, one has $\sum_{i=1}^m \alpha_i \geq \left(\sum_{i=1}^m \alpha_i^2\right)^{\frac{1}{2}}$, we deduce that for all $y\geq 0$, $z\geq 0$, we have

$$
\begin{aligned}
\tilde{\Phi}(w) \quad &\geq \quad c\sum_{i=1}^m \min(y_i,z_i) \\
&\geq \quad c\left(\sum_{i=1}^m \min(y_i,z_i)^2\right)^{\frac{1}{2}} \\
&= \quad c\,\mathrm{dist}(w,\Delta).
\end{aligned}
$$

Therefore hypothesis $\mathcal{H}_2$ is satisfied by $\tilde{\Phi}$ with the same constant c as for Φ. Hypothesis $\mathcal{H}_3$ follows from the concavity of the function $(a,b)\to\min(a,b)$.

Hypothesis $\mathcal{H}_4$ follows from the fact that $-\Phi$ is a polyhedral convex function; thus, its subdifferential is uniformely bounded.

In order to prove the inclusion $\partial(-\tilde{\Phi})(\overline{w})\subset\partial(-\Phi)(\overline{w})$, we compute the subdifferential $\partial(-\tilde{\Phi})(\overline{w})$, and obtain that $\partial(-\tilde{\Phi})(\overline{w})$ is the convex hull of the set $\{w^*,\overline{w}^*\}$. Given that the set $\partial(-\Phi)(\overline{w})$ is convex and contains the set $\{w^*,\overline{w}^*\}$, we then have $\partial(-\tilde{\Phi})(\overline{w})\subset\partial(-\Phi)(\overline{w})$. □

The above lemma permits us to prove the proposition below.

Proposition 5.3. *We suppose f is differentiable on Ω. Let a function Φ satisfy hypotheses $\mathcal{H}_1$-$\mathcal{H}_4$, and c be a constant satisfying assumption $\mathcal{H}_2$. Let $\mu>\frac{L}{c\alpha}$, where α is given by* (4.2). *Let $\overline{w}$ be a feasible point for MPCC* (2.1). *Assume that the set*

$$
\left\{\left((y_i^*)_{i\in I_y(\overline{w})\setminus I_z(\overline{w})},(z_i^*)_{i\in I_z(\overline{w})\setminus I_y(\overline{w})}\right)\ \mid\ w^*:=(x^*,y^*,z^*)\in\partial(-\Phi)(\overline{w})\right\}
$$

is a singleton. The point $\overline{w}$ is a solution of the optimization problem

$$
\begin{array}{ll}
\min & \langle\nabla f(\overline{w}),w\rangle \\
\textit{subject to} & w:=(x,y,z)\ ,\ g(w)\leq 0\ ,\ h(w)=0 \\
 & y\geq 0\ ,\ z\geq 0 \\
 & \langle y,z\rangle=0.
\end{array}
\tag{5.3}
$$

if and only if the inclusion holds:

$$
\partial(-\mu\Phi)(\overline{w})\subset\nabla f(\overline{w})+N_\Omega(\overline{w}).
$$

Proof. Given that $\|\nabla f(\overline{w})\|\leq L$, the function $w\to\langle\nabla f(\overline{x}),w\rangle$ is Lipschitz-continuous with a constant of Lipschitz L. Observe that both MPCC (2.1) and (5.3) have the same constraint set which can be written as $\Omega\cap\Delta$, where Ω and Δ are defines in (2.2) and (2.3).

Suppose that $\overline{w}$ is a solution of the MPCC (5.3). According to Proposition 4.1, $\overline{w}$ is a solution of

$$
\begin{array}{ll}
\min & \langle\nabla f(\overline{w}),w\rangle+\mu\Phi(w) \\
\text{subject to} & w\in\Omega,
\end{array}
$$

which can be written as follows:

$$
\min_{w\in\mathbb{R}^n}\langle\nabla f(\overline{w}),w\rangle+\delta_\Omega(w)-(-\mu\Phi(w))
\tag{5.4}
$$

That is a DC program. According to Proposition 3.1, we have:

$$\partial(-\mu\Phi)(\overline{w}) \subset \partial\left(\langle \nabla f(\overline{w}), \cdot \rangle + \delta_\Omega(\cdot)\right)(\overline{w}) = \nabla f(\overline{x}) + N_\Omega(\overline{w}).$$

Conversely, suppose that $\partial(-\mu\Phi)(\overline{w}) \subset \nabla f(\overline{w}) + N_\Omega(\overline{w})$. According to Lemma 5.5, there exists a polyhedral concave function $-\tilde{\Phi}$ which satisfies the hypotheses $\mathcal{H}_1, \cdots, \mathcal{H}_4$ with the same constant $c > 0$ in $\mathcal{H}_2$ and which satisfies:

$$\partial(-\tilde{\Phi})(\overline{w}) \subset \partial(-\Phi)(\overline{w}).$$

We then have $\partial(-\mu\tilde{\Phi})(\overline{w}) \subset \nabla f(\overline{w}) + N_\Omega(\overline{w})$. Given that $-\tilde{\Phi}$ is a polyhedral and convex function, by Proposition 3.1, $\overline{w}$ is a solution of:

$$\min_{w \in \mathbb{R}^n} \langle \nabla f(\overline{w}), w \rangle + \delta_\Omega(w) - (-\mu\tilde{\Phi}(w))$$

According to Lemma 5.5, the function $\tilde{\Phi}$ satisfies assumptions $\mathcal{H}_1, \cdots, \mathcal{H}_4$, then by Proposition 4.1, $\overline{w}$ is a solution of

$$\begin{array}{ll} \min & \langle \nabla f(\overline{w}), w \rangle \\ \text{subject to} & w := (x, y, z)\ ,\ g(w) \leq 0\ ,\ h(w) = 0 \\ & y \geq 0\ ,\ z \geq 0 \\ & \langle y, z \rangle = 0. \end{array}$$

□

Theorem 5.4. *We suppose f differentiable on Ω. We suppose f is differentiable on Ω. Let a function Φ satisfy hypotheses $\mathcal{H}_1$-$\mathcal{H}_4$, and c be a constant satisfying assumption $\mathcal{H}_2$. Let $\mu > \frac{L}{c\alpha}$, where α is given by (4.2). Let $\overline{w} \in \Omega$ be a feasible point for MPCC (2.1). Assume that the set*

$$\left\{ \left((y_i^*)_{i \in I_y(\overline{w}) \setminus I_z(\overline{w})}, (z_i^*)_{i \in I_z(\overline{w}) \setminus I_y(\overline{w})} \right) \mid w^* := (x^*, y^*, z^*) \in \partial(-\Phi)(\overline{w}) \right\}$$

is a singleton. If $\overline{w}$ is a strongly stationary point for MPCC, then the following inclusion holds:

$$\partial(-\mu\Phi)(\overline{w}) \subset \nabla f(\overline{w}) + N_\Omega(\overline{w}).$$

If moreover MPCC-LICQ holds at $\overline{w}$, then the converse holds truly.

Proof. We first suppose that $\overline{w}$ is a strongly stationary point. By Theorem 2.6, 0 is a solution of the following problem where d is a decision variable:

$$\begin{array}{ll} \min & \langle \nabla f(\overline{w}), d \rangle \\ \text{subject to} & g(\overline{w}) + \nabla g(\overline{w})d \leq 0\ ,\ h(\overline{w}) + \nabla h(\overline{w})d = 0 \\ & 0 \leq \overline{y} + d_1 \perp \overline{z} + d_2 \geq 0. \end{array}$$

By convexity of the functions g_i, one has $\{d \in \mathbb{R}^n \mid g(\overline{w} + d) \leq 0\} \subset \{d \in \mathbb{R}^n \mid g(\overline{w}) + \nabla g(\overline{w})d \leq 0\}$. Moreover, since h is an affine function and $h(\overline{w}) = 0$, one has $h(\overline{w} + d) = \nabla h(\overline{w})d$. Therefore 0 solves the following problem where d is a decision variable:

$$\begin{array}{ll} \min & \langle \nabla f(\overline{w}), d \rangle \\ \text{subject to} & g(\overline{w} + d) \leq 0\ ,\ h(\overline{w} + d) = 0 \\ & 0 \leq \overline{y} + d_1 \perp \overline{z} + d_2 \geq 0. \end{array}$$

According to Proposition 5.3, the inclusion $\partial(-\mu\Phi)(\overline{w}) \subset \nabla f(\overline{w}) + N_\Omega(\overline{w})$ holds.

Conversely, we suppose that $\partial(-\mu\Phi)(\overline{x}) \subset \nabla f(\overline{w}) + N_\Omega(\overline{w})$ and MPCC-LICQ holds. By Proposition 5.3, $\overline{w}$ is a solution of the MPCC

$$\begin{array}{ll} \min & \langle \nabla f(\overline{w}), w \rangle \\ \text{subject to} & g(w) \leq 0\ ,\ h(w) = 0 \\ & 0 \leq y \perp z \geq 0. \end{array}$$

According to Theorem 2.4, $\overline{w}$ is a strongly stationary point for MPCC since MPCC-LICQ holds at $\overline{w}$. □

We can observe that the inclusion $\partial(-\mu\Phi)(\overline{w}) \subset \nabla f(\overline{w}) + N_\Omega(\overline{w})$ uses only convex analysis tools because $-\mu\Phi$ is a convex function and Ω is a convex set, though the constraint sets of the MPCCs are not convex in general.

Another notion of stationary point for the DC program $\min_{w \in W} u(w) - v(w)$, with u and v convex functions, is that $\partial u(\overline{w}) \cap \partial v(\overline{w}) \neq \emptyset$. This notion applied to the DC reformulation of MPCC leads to:

$$\partial(-\mu\Phi)(\overline{w}) \cap (\nabla f(\overline{w}) + N_\Omega(\overline{w})) \neq \emptyset.$$

The following proposition shows that the above condition is related to the weakly stationarity of MPCC at $\overline{w}$. Before proposition we need a lemma:

Lemma 5.5. *Let w a feasible point for MPCC, and $w^* := (x^*, y^*, z^*) \in \partial(-\Phi)(w)$. Then for all $i \notin I_y(w)$, one has $y_i^* = 0$ and for all $i \notin I_z(w)$, one has $z_i^* = 0$.*

Proof. Let $i \notin I_y(w)$ and $w^* := (x^*, y^*, z^*) \in \partial(-\Phi)(w)$. Let y_i' in a neighborhood of y_i. We consider $w' := (x, (y_1, \cdots, y_{i-1}, y_i', y_{i+1}, \cdots, y_m), z)$. Let $x' \in \mathbb{R}^n$, $y', z' \in \mathbb{R}^m$ be such that $w' := (x', y', z')$. We have $y' \geq 0$, $z' \geq 0$ and $\langle y', z' \rangle = 0$ since $z_i = 0$ and $y_i > 0$. Therefore, one has $\Phi(w') = 0$, for all y_i' in a neighborhood of y_i, which ensure that $y_i^* = 0$. In the same way, if $i \notin I_z(w)$, one has $z_i^* = 0$. □

We now gives a sufficient condition for a feasible point of MPCC to be weakly stationary.

Proposition 5.6. *We suppose f is differentiable on Ω and Lipchitz-continuous with $L \geq 0$ a constant of Lipschitz. Let a function Φ satisfy hypotheses $\mathcal{H}_1$-$\mathcal{H}_4$, and c be a constant satisfying assumption $\mathcal{H}_2$. Let $\mu > \frac{L}{c\alpha}$, where α is given by (4.2). Let $\overline{w}$ a feasible point for MPCC (2.1). If $\overline{w}$ satisfies the following property*

$$\partial(-\mu\Phi)(\overline{w}) \cap (\nabla f(\overline{w}) + N_\Omega(\overline{w})) \neq \emptyset,$$

and the constraint set Ω is qualified at $\overline{w}$ (see Definition 2.1), then $\overline{w}$ is a weakly stationary point for MPCC.

Proof. Since $\partial(-\mu\Phi)(\overline{w}) \cap (\nabla f(\overline{w}) + N_\Omega(\overline{w})) \neq \emptyset$, there exists an element $-\overline{w}^* := -(\overline{x}^*, \overline{y}^*, \overline{z}^*) \in \partial(-\Phi)(\overline{w})$ such that $-\mu\overline{w}^* \in \nabla f(\overline{w}) + N_\Omega(\overline{w})$. Therefore one has

$$\nabla f(\overline{w}) + \mu\overline{w}^* \in -N_\Omega(\overline{w}).$$

Therefore, since the constraint set Ω is qualified at $\overline{w}$, there exist Lagrange multipliers λ^g, λ^h, ν_1, ν_2 such that

$$\nabla f(\overline{w}) +^t \nabla g(\overline{w})\lambda^g +^t \nabla h(\overline{w})\lambda^h - \begin{pmatrix} 0 \\ \nu_1 - \mu\overline{y}^* \\ \nu_2 - \mu\overline{z}^* \end{pmatrix} = 0$$

and

$$0 \geq g(\overline{w}) \perp \lambda^g \geq 0 \ , \ 0 \leq \nu_1 \perp \overline{y} \geq 0 \ , \ 0 \leq \nu_2 \perp \overline{z} \geq 0.$$

Let the vector $\lambda = (\lambda^g, \lambda^h, \hat{\nu}_1, \hat{\nu}_2)$ defined by $(\hat{\nu}_{1,i}, \hat{\nu_{2,i}}) = (\nu_1 - \mu\overline{y}_i^*, \nu_2 - \mu\overline{z}_i^*)$. Let $i \notin I_y(\overline{w})$, since $\overline{w} \in \Delta$, we have $\overline{z}_i = 0$, moreover since $\overline{y}_i > 0$, one has $\overline{y}_i^* = 0$ by Lemma (5.5). Moreover, we have $\nu_{1,i} = 0$ by complementarity condition, which implies that $\hat{\nu}_{1,i} = 0$. In the same way, if $\overline{z}_i > 0$, then $\hat{\nu}_{2,i} = 0$. That implies that $(\overline{w}, \lambda)$ is a KKT stationary point of the TNLP (2.6), thus $\overline{w}$ is a weakly stationary point for MPCC. □

6 DCA Algorithm for MPCC

In this section we apply the DC method, which has been introduced by Pham Dinh Tao and Le Thi [30, 31, 32, 33], to MPCC.

We recall that the MPCC that we consider, given in (2.1), can be written as follows:

$$\begin{array}{ll} \min & f(w) \\ \text{subject to} & w \in \Omega \\ & \langle y, z \rangle = 0, \end{array}$$

where the set Ω is given by (2.2). We suppose that $\alpha > 0$ where α is defined by (4.2), and consider a function Φ which satisfies hypotheses $\mathcal{H}_1, \cdots, \mathcal{H}_4$, where the hypotheses $\mathcal{H}_i$ are given in Section 4. We suppose that f is L-continuous Lipschitz on Ω. Let $c > 0$ be the constant given by hypothesis $\mathcal{H}_2$. As we have seen in Section 4, for any $\mu > \dfrac{L}{c\alpha}$, MPCC (2.1) is equivalent to

$$\begin{array}{ll} \min & f(w) + \mu\Phi(w) \\ \text{subject to} & w \in \Omega \end{array}$$

Consider the following DC descomposition of f:

$$f := f_1 - f_2, \tag{6.1}$$

with f_1 and f_2 two convex functions on Ω. That is, the MPCC is equivalent to the DC program

$$\min_{x \in \mathbb{R}^n} (f_1 + \delta_\Omega)(w) - (f_2 - \mu\Phi)(w). \tag{6.2}$$

where we recall that $\delta_\Omega(w) = \left\{ \begin{array}{ccc} 0 & \text{if} & w \in \Omega \\ +\infty & \text{if} & w \notin \Omega \end{array} \right.$.

The DC algorithm (abbreviated as DCA), starting from an initial point $w^0 \in \Omega$, constructs two sequences $(w^k)_k$ and $(v^k)_k$ by

$$v^k \in \partial(f_2 - \mu\Phi)(w^k), \tag{6.3}$$

and

$$w^{k+1} \in \partial(f_1 + \delta_\Omega)^*(v^k). \tag{6.4}$$

Remark 6.1. If $v^k \to \overline{v}$ and $w^k \to \overline{w}$, given that the subdifferential of a lower semi continuous function has closed graph, we have $\overline{v} \in \partial(f_2 - \mu\Phi)(\overline{w})$ and $\overline{w} \in \partial(f_1 + \delta_C)^*(\overline{v})$. Since $\overline{w} \in \partial(f_1 + \delta_C)^*(\overline{v}) \Leftrightarrow \overline{v} \in \partial(f_1 + \delta_C)(\overline{w})$, we deduce that $\partial(f_1 + \delta_\Omega)(\overline{w}) \cap \partial(f_2 - \mu\Phi)(\overline{w}) \neq \emptyset$, which is equivalent, under assumption of differentiability of f_1 and f_2, to

$$\partial(-\mu\Phi)(\overline{w}) \cap (\nabla f(\overline{w}) + N_\Omega(\overline{w})) \neq \emptyset.$$

Therefore, according to Proposition 5.6, all limit $\overline{w} := (\overline{x}, \overline{y}, \overline{z})$ of (w^k) is a weakly stationary point for MPCC if Ω is qualified at $\overline{w}$ and $\langle \overline{y}, \overline{z} \rangle = 0$.

6.1 A regularized scheme of DCA

Throught this subsection we assume that the functions f_1 and f_2 are differentiable on Ω, where f_1 and f_2 satisfy (6.1). The DCA constructs the sequences (v^k) and (w^k) by (6.3) and by (6.4). If Φ is differentiable at w^k, we then have:

$$v^k = \nabla(f_2 - \mu\Phi)(w^k).$$

Therefore, we have $w^{k+1} \in \partial(f_1 + \delta_\Omega)^* \Big(\nabla(f_2 - \mu\Phi)(w^k) \Big)$ which implies that:

$$\nabla(f_2 - \mu\Phi)(w^k) \in \partial(f_1 + \delta_\Omega)(w^{k+1}).$$

The above statement is equivalent to w^{k+1} is a solution of the following convex optimization problem, which consists of linearizing the concave part of (6.2):

$$\begin{array}{ll} \min & f_1(w) - \langle \nabla f_2(w^k) - \mu\nabla\Phi(w^k), w \rangle \\ \text{subject to} & w \in \Omega. \end{array}$$

As you can see, if Φ is differentiable at w^k, then w^{k+1} can be obtained from w^k solving a convex optimization problem. But in general, Φ is not differentiable at w^k. We consider the family of functions Φ_ρ which are defined as follows:

$$\forall \rho \geq 0,\ \forall w \in \Omega\ ,\Phi_\rho(w) := \sum_{i=1}^{m} \left(y_i + z_i + \rho - \sqrt{y_i^2 + z_i^2 + \rho^2} \right). \tag{6.5}$$

You can observe that for any $\rho > 0$, the function Φ_ρ is differentiable on $\mathbb{R}^n$. The DC Algorithm regularized scheme consists of, given the vector w^k, choosing a real $\rho_k > 0$ and computing the next iterate w^{k+1} as a solution of this following convex optimization problem:

$$\begin{array}{ll} \min & f_1(w) - \langle \nabla f_2(w^k) - \mu\nabla\Phi_{\rho_k}(w^k), w \rangle \\ \text{subject to} & w \in \Omega. \end{array} \tag{6.6}$$

We call this algorithm "DC Algorithm for MPCC" (DCA-MPCC). We give some properties about the function Φ_ρ defined above (6.5). First we introduce the following function $\theta : \mathbb{R}^3 \to \mathbb{R}$.

Proposition 6.2. *Let $\theta : \mathbb{R}^3 \to \mathbb{R}$ defined by $\theta(a, b, \rho) := a + b + \rho - \sqrt{a^2 + b^2 + \rho^2}$. The function θ satisfies the following properties:*

1. *For all $a, b \in \mathbb{R}_+$, $\theta(a, b, 0) \geq \frac{2}{2+\sqrt{2}} \min\{a, b\}$.*
2. *The function $\theta(\cdot, \cdot, 0)$ is differentiable for all $(a, b) \in (\mathbb{R}_+)^2 \setminus \{(0, 0)\}$, and $\partial_{(a,b)}(-\theta(\cdot, \cdot, 0))(0, 0) = B((-1, -1), 1)$.*
3. *The function θ is Lipschitz-continuous on $\mathbb{R}^3$.*
4. *For all $\rho \geq 0$, $\theta(\cdot, \cdot, \rho)$ is concave on $\mathbb{R}^2$.*

Proof. 1. Let $a, b \in \mathbb{R}$ such that $0 < a \leq b$. We have:

$$\begin{aligned}
\theta(a,b,0) &= a+b-\sqrt{a^2+b^2} \\
&= \frac{2ab}{a+b+\sqrt{a^2+b^2}} \\
&\geq \frac{2ab}{2b+\sqrt{2b^2}} \\
&= \frac{2}{2+\sqrt{2}}a \\
&= \frac{2}{2+\sqrt{2}}\min\{a,b\}.
\end{aligned}$$

If $0 < b \leq a$ we obtain the result by the same calculus. If $a = 0$ and $b \geq 0$ or $b = 0$ and $a \geq 0$, then $\theta(a,b,0) = 0 = \min\{a,b\}$, that proves the first result of the Proposition.

2. It is clear that $\theta(\cdot,\cdot,0)$ is differentiable over $(\mathbb{R}_+)^2 \setminus \{(0,0)\}$. The equality $\partial_{(a,b)}(-\theta(0,0,0)) = B((-1,-1),1)$ is deduced by the equality $\partial\|\cdot\|(0) = B(0,1)$, with $\|\cdot\|$ an euclidian norm.
3. It is easy to verify that for all $(a,b,\rho) \in \mathbb{R}^3$, for all $x^* \in \partial\theta(a,b,\rho)$, one has $\|x^*\| \leq 3\sqrt{2}$, which ensures that θ is Lipschitz-continuous on $\mathbb{R}^3$.
4. We can observe that $\sqrt{a^2+b^2+\rho^2} = \|(a,b,\rho)\|$ which is convex with respect to its three variables, then it is convex with respect to (a,b). Therefore, for all $\rho \geq 0$, $\theta(\cdot,\cdot,\rho)$ is a concave function because it is the difference between a linear function and a convex function.

□

Since we have $\Phi_\rho(w) = \sum_{i=1}^{m} \theta(y_i, z_i, \rho)$, we can deduce the following properties about the function $(\rho, w) \to \Phi_\rho(w)$

Proposition 6.3. *The function $(\rho, w) \to \Phi(w, \rho)$ satisfies the following properties:*

1. *There exists a constant $c > 0$ such that for all $w \in \Omega$, $\Phi_0(w) \geq c dist(w, \Delta)$.*
2. *For all $w \in \Omega$, $w^* \in \partial(-\Phi_0)(w)$ if and only if $(y_i^*, z_i^*) = -\nabla_{y_i,z_i}\theta(y_i, z_i, 0)$ if $(y_i, z_i) \neq (0,0)$ and $(y_i^*, z_i^*) \in B((-1,-1),1)$ if $(y_i, z_i) = (0,0)$.*
3. *The function $(\rho, w) \to \Phi_\rho(w)$ is Lipschitz-continuous on $\mathbb{R}^n$.*
4. *For all $\rho \geq 0$, Φ_ρ is concave on $\mathbb{R}^n$.*

Proof. It is a direct consequence of Proposition 6.2 given that $\Phi_\rho(w) = \sum_{i=1}^{m} \theta(y_i, z_i, \rho)$. □

From the previous proposition, we deduce the following corollary.

Corollary 6.4. *The function Φ_0 given by (6.5) satisfies hypotheses $\mathcal{H}_1, \cdots, \mathcal{H}_4$ given in Section 4.*

Proof. It is a direct consequence of Proposition 6.3. □

The following theorem gives a convergence result for DCA-MPCC.

Theorem 6.5. *We suppose that f is a C^1 and a Lipchitz-continuous function with modulus $L \geq 0$ on Ω and Ω is a bounded set. We suppose that $f = f_1 - f_2$ with f_1 and f_2 two C^1 and convex functions on Ω. Moreover, we suppose that f_1 or f_2 is γ-strongly convex on Ω, with $\gamma > 0$. We suppose that $\alpha > 0$ where α is defined in (4.2). We choose Φ_ρ defined in (6.5) and let $c > 0$ be the constant of assumption $\mathcal{H}_2$ for Φ_0. We construct a sequence $(w^k)_k$ with $w^0 \in \Omega$ and w^{k+1} is a solution of (6.6) with $\mu > \dfrac{L}{c\alpha}$ and ρ_k chosen such that $\rho_k^2 = o(\min\{(y_i^k)^2 + (z_i^k)^2 \mid i = 1, \cdots, m\})$, $\rho_k \to 0$ and $\sum_{k \geq 0} |\rho_{k+1} - \rho_k| < \infty$. The following statements hold:*

1. *We have $\|w^{k+1} - w^k\| \to 0$.*

2. *Any limit $\overline{w} := (\overline{x}, \overline{y}, \overline{z})$ of the sequence (w^k) satisfie*
$$\partial(-\mu\Phi_0)(\overline{w}) \cap (\nabla f(\overline{w}) + N_\Omega(\overline{w})) \neq \emptyset.$$
If moreover $\langle \overline{y}, \overline{z}\rangle = 0$ and Ω is qualified at $\overline{w}$ (Definition 2.1), then $\overline{w}$ is a weakly stationary point for MPCC.
3. *Any limit $\overline{w} := (\overline{x}, \overline{y}, \overline{z})$ of the sequence (w^k) satisfies*
$$f(\overline{w}) + \mu\Phi_0(\overline{w}) \leq f(w^0) + \mu\Phi_0(w^0) + M\sum_{k\geq 0} |\rho_{k+1} - \rho_k|$$
where M is the constant of Lipschitz of the function $(\rho, w) \to \Phi_\rho(w)$.
4. *If Ω is bounded and the set*
$$\{w \in \Omega \mid \partial(-\mu\Phi_0)(w) \cap (\nabla f(w) + N_\Omega(w)) \neq \emptyset\}$$
is finite, then the whole sequence (w^k) converges.

Proof. We prove each statement.

1. If there exist $k_0 \in \mathbb{N}$ such that $w^{k_0} = w^{k_0+1}$ then $w^k = w^{k_0}$ for any $k \geq k_0$. We suppose that for all $k \in \mathbb{N}$, one has $w^k \neq w^{k+1}$.

   If f_1 is γ-strongly convex, then one has
$$f_1(w^{k+1}) \leq f_1(w^k) + \langle \nabla f_1(w^{k+1}), w^{k+1} - w^k\rangle - \frac{\gamma}{2}\|w^{k+1} - w^k\|^2,$$
   or else by convexity of f_1, one has
$$f_1(w^{k+1}) \leq f_1(w^k) + \langle \nabla f_1(w^{k+1}), w^{k+1} - w^k\rangle.$$
   If f_1 is not γ-strongly convex, then f_2 is γ-strongly convex, and since $-\mu\Phi_{\rho_k}$ is convex, we obtain
$$\begin{aligned} f_2(w^{k+1}) - \mu\Phi_{\rho_k}(w^{k+1}) \quad &\geq \quad f_2(w^k) - \mu\Phi_{\rho_k}(w^k) + \frac{\gamma}{2}\|w^{k+1} - w^k\|^2 \\ &+ \quad \langle \nabla f_2(w^k) - \mu\nabla\Phi_{\rho_k}(w^k), w^{k+1} - w^k\rangle. \end{aligned}$$
   If f_2 is not γ-strongly convex, it is only convex (and f_1 is γ-strongly convex), thus one has
$$f_2(w^{k+1}) - \mu\Phi_{\rho_k}(w^{k+1}) \geq f_2(w^k) - \mu\Phi_{\rho_k}(w^k) + \langle \nabla f_2(w^k) - \mu\nabla\Phi_{\rho_k}(w^k), w^{k+1} - w^k\rangle.$$
   In both cases, the following inequalities hold:
$$\begin{aligned} f_1(w^{k+1}) - (f_2(w^{k+1}) - \mu\Phi_{\rho_k}(w^{k+1})) \quad &\leq \quad f_1(w^k) - (f_2(w^k) - \mu\Phi_{\rho_k}(w^k)) \\ &- \quad \frac{\gamma}{2}\|w^{k+1} - w^k\|^2 + \langle \nabla f_1(w^{k+1}) \\ &- \quad (\nabla f_2(w^k) - \mu\nabla\Phi_{\rho_k}(w^k)), w^{k+1} - w^k\rangle \\ &\leq \quad f_1(w^k) - (f_2(w^k) - \mu\Phi_{\rho_k}(w^k)) \\ &- \quad \frac{\gamma}{2}\|w^{k+1} - w^k\|^2. \end{aligned}$$
   The last inequality holds true because w^{k+1} solves problem (6.6), thus $\nabla f_1(w^{k+1}) - (\nabla f_2(w^k) - \mu\nabla\Phi_{\rho_k}(w^k)) \in -N_\Omega(w^{k+1})$.

   Finally, using the Lipschitz-continuity of $(\rho, w) \to \Phi_\rho(w)$ (which it is ensured by Proposition 6.3), one has (with $M \geq 0$ a constant of Lipschitz-continuity of $(\rho, w) \to \Phi_\rho(w)$):
$$\begin{aligned} \frac{\gamma}{2}\sum_{k=0}^{p-1}\|w^{k+1} - w^k\|^2 \quad &\leq \quad f_1(w^0) - f_2(w^0) - f_1(w^p) + f_2(w^p) \\ &+ \quad \mu\Phi_{\rho_p}(w^p) - \mu\Phi_{\rho_0}(w^0) + M\sum_{k=0}^{p-1}|\rho_{k+1} - \rho_k| \\ &= \quad f(w^0) - f(w^p) + \mu\Phi_{\rho_0}(w^0) - \mu\Phi_{\rho_0}(w^p) \\ &+ \quad M\sum_{k=0}^{p-1}|\rho_{k+1} - \rho_k|. \end{aligned} \tag{6.7}$$
   Since the function f and Φ_{ρ_0} are bounded from below and the serie $\sum|\rho_{k+1} - \rho_k|$ is convergent, we deduce that the serie $\sum\|w^{k+1} - w^k\|^2$ is also convergent, thus $\|w^{k+1} - w^k\| \to 0$.

2. Take a subsequence (w^{k_j}) such that $w^{k_j} \to \overline{w}$. According to Proposition 6.3, the sequence $\left(\left\|\nabla\Phi_{\rho_k}(w^{k_j})\right\|\right)_j$ is bounded, then there exists a subsequence $(w^{k_{j_l}})_l$ such that $\nabla\Phi_{\rho_{k_{j_l}}}(w^{k_{j_l}}) \longrightarrow \overline{w}^*$. We show that $-\overline{w}^* \in \partial(-\Phi_0)(\overline{w})$.
Let $i \in \{1, \cdots, m\}$ such that $\overline{y}_i + \overline{z}_i > 0$. One has

$$\begin{aligned}
\frac{\partial \Phi_{\rho_{k_{j_l}}}}{\partial y_i}(w^{k_{j_l}}) &= 1 - \frac{y_i^{k_{j_l}}}{\sqrt{(y_i^{k_{j_l}})^2 + (z_i^{k_{j_l}})^2 + \rho_{k_{j_l}}^2}} \\
&\to 1 - \frac{\overline{y}_i}{\sqrt{\overline{y}_i^2 + \overline{z}_i^2}} \text{ because } \rho_k \to 0 \\
&= \frac{\partial \Phi_0}{\partial y_i}(\overline{w}).
\end{aligned}$$

In the same way, $\dfrac{\partial \Phi_{\rho_{k_{j_l}}}}{\partial z_i}(w^{k_{j_l}}) \longrightarrow \dfrac{\partial \Phi_0}{\partial z_i}(\overline{w})$. We suppose that $\overline{y}_i^2 + \overline{z}_i^2 = 0$. Then one has:

$$\begin{aligned}
(1-\overline{y}_i^*)^2 + (1-\overline{z}_i^*)^2 &= \lim_{k\to+\infty} \left(1 - \frac{\partial \Phi}{\partial y_i}(w^{k_{j_l}}, \rho_{k_{j_l}})\right)^2 + \left(1 - \frac{\partial \Phi}{\partial z_i}(w^{k_{j_l}}, \rho_{k_{j_l}})\right)^2 \\
&= \lim_{k\to+\infty} \frac{(y_i^{k_{j_l}})^2 + (z_i^{k_{j_l}})^2}{(y_i^{k_{j_l}})^2 + (z_i^{k_{j_l}})^2 + \rho_{k_{j_l}}^2} \\
&= 1 - \lim_{k\to+\infty} \frac{\rho_{k_{j_l}}^2}{(y_i^{k_{j_l}})^2 + (z_i^{k_{j_l}})^2 + \rho_{k_{j_l}}^2} \\
&= 1 \text{ because } \rho_{k_{j_l}}^2 = o((y_i^{k_{j_l}})^2 + (z_i^{k_{j_l}})^2).
\end{aligned}$$

Therefore we have $-(\overline{y}_i^*, \overline{z}_i^*) \in B((-1,-1),1)$. According to Proposition 6.3, we finally have $-\overline{w}^* \in \partial_x(-\Phi_0)(\overline{w})$.

Since for all $j \in \mathbb{N}$, w^{k_j+1} solves the optimization problem (6.6), one has $\nabla f_1(w^{k_j+1}) - \nabla f_2(w^{k_j}) + \mu\nabla\Phi_{\rho_{k_j}}(w^{k_j}) \in -N_\Omega(w^{k_j})$. From Item 1 of this theorem, we have:

$$\|w^{k_j+1} - \overline{w}\| \leq \|w^{k_j+1} - w^{k_j}\| + \|w^{k_j} - \overline{w}\| \to 0.$$

We deduce that $w^{k_j+1} \to \overline{w}$. Since f_1 and f_2 are C^1 functions on Ω, N_Ω has a closed graph and $\nabla\Phi_{\rho_{k_j}}(w^{k_j}) \to \overline{w}^*$, we obtain:

$$\nabla f(\overline{w}) + \mu\overline{w}^* \in -N_\Omega(\overline{w})$$

which implies that

$$-\mu\overline{w}^* \in \nabla f(\overline{w}) + N_\Omega(\overline{w}).$$

Since at the same time, $-\mu\overline{w}^* \in \partial(-\mu\Phi_0)(\overline{w})$, we deduce that:

$$-\mu\overline{w}^* \in \partial(-\mu\Phi_0)(\overline{w}) \cap (\nabla f(\overline{w}) + N_\Omega(\overline{w})),$$

which implies that $\partial(-\mu\Phi_0)(\overline{w}) \cap (\nabla f(\overline{w}) + N_\Omega(\overline{w})) \neq \emptyset$.

According to Proposition 5.6, if $\langle \overline{y}, \overline{z} \rangle = 0$ and Ω is qualified at $\overline{w}$, then $\overline{w}$ is a weakly stationary point for MPCC.

3. The inequalities (6.7) imply that:

$$\begin{aligned}
f(w^p) + \mu\Phi_0(w^p) &\leq f(w^0) + \mu\Phi_0(w^0) + M\sum_{k=0}^{p-1} |\rho_{k+1} - \rho_k| \\
&\leq f(w^0) + \mu\Phi_0(w^0) + M\sum_{k=0}^{+\infty} |\rho_{k+1} - \rho_k|.
\end{aligned}$$

We deduce that if $\overline{w}$ is a limit of w^p, then we have:

$$f(\overline{w}) + \mu\Phi_0(\overline{w}) \leq f(w^0) + \mu\Phi_0(w^0) + M\sum_{k=0}^{+\infty} |\rho_{k+1} - \rho_k|.$$

4. The set of limits of the sequence (w^k) is nonempty because Ω is a bounded set. It is finite because by item 2 of this theorem, it is included in the set

$$\{w \in \Omega \mid \partial(-\mu\Phi_0)(w) \cap (\nabla f(w) + N_\Omega(w)) \neq \emptyset\}$$

which is finite by assumption. Moreover, item 1 of this theorem implies that the set of limits of the sequence (w^k) is connex. Finally, the set of limits of the sequence (w^k) is a singleton, then the whole sequence (w^k) converges.

□

The convergence to a weakly stationary point is not satisfactory because in general it is easy to obtain weakly stationary points. The following algorithm allows for obtaining strongly stationary points if MPCC-LICQ holds. Before, for any $\overline{w} := (\overline{x}, \overline{y}, \overline{z}) \in \Omega$, we introduce the following function $\tilde{\Phi}_{\overline{w}} : \mathbb{R}^n \to \mathbb{R}$ with, for all $w := (x, y, z) \in \Omega$:

$$\tilde{\Phi}_{\overline{w}}(w) := \sum_{\substack{i=1 \\ \overline{y}_i + \overline{z}_i > 0}}^{m} \left(y_i + z_i - \sqrt{y_i^2 + z_i^2} \right) + \sum_{\substack{i=1 \\ \overline{y}_i + \overline{z}_i = 0}}^{m} \min\{y_i, z_i\}. \tag{6.8}$$

Difference of Convex functions Algorithm Completed for MPCC (DCAC-MPCC)

Step 0: Choose $w^0 \in \Omega$ and $\varepsilon^0 > 0$. Set $p = 0$.

Step 1: Construct a sequence (w^k) by w^{k+1} solving (6.6). The parameters ρ_k are chosen such that $\rho_k^2 = o(\min\{(y_i^k)^2 + (z_i^k)^2 \mid i = 1, \cdots, m\})$, $\rho_k \to 0$ and $\sum_{k \geq 0} |\rho_{k+1} - \rho_k| < \varepsilon^p / M$, where M is a constant Lipschitz of $(\rho, w) \to \Phi_\rho(w)$.

Step 2: Let $\overline{w}^p$ the limit of the sequence (w^k). If for all $w^* \in \mathrm{Ext}(\partial(-\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p))$, $\overline{w}^p$ is a solution of

$$\min_{w \in \Omega} \frac{1}{2}\|w\|^2 + \langle \nabla f(\overline{w}^p) - \mu w^* - \overline{w}^p, w \rangle. \tag{6.9}$$

then STOP, else go to Step 3.

Step 3: Pick $w^* \in \mathrm{Ext}(\partial(-\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p))$ be such that $\overline{w}^p$ is not a solution of (6.9), set $\tilde{w}^p$ as a solution of (6.9) and go to Step 1 with $w^0 := \tilde{w}^p$, $\varepsilon^{p+1} := f(\overline{w}^p) + \mu\Phi_0(\overline{w}^p) - f(\tilde{w}^p) - \mu\Phi_0(\tilde{w}^p) > 0$ and $p \to p + 1$.

The notation $\mathrm{Ext}(\partial(-\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p))$ stands for the extremal points for the convex set $\partial(-\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p)$. Since $\partial(-\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p)$ is a polyhedral set, the set $\mathrm{Ext}(\partial(-\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p))$ is finite, then Step 2 consists of solving a finite number of optimization problems. You can observe that (6.9) has a unique solution. Steps 2 and 3 have been inspired by Section 5.1 in [13].

The following theorem gives a convergence result of DCAC-MPCC.

Theorem 6.6. *We suppose that f is a C^1 and a Lipchitz-continuous function on Ω with $L \geq 0$ its constant of Lipschitz, that Ω is a bounded set, and that $f = f_1 - f_2$ with f_1 and f_2 two C^1 and convex functions on Ω. Moreover, we suppose that f_1 or f_2 is γ-strongly convex on Ω, with $\gamma > 0$. We suppose that $\alpha > 0$ where α is defined in (4.2). We choose Φ_ρ defined in (6.5) and let $c > 0$ be the constant of assumption $\mathcal{H}_2$ for Φ_0. If the set*

$$\{w \in \Omega \mid \partial(-\mu\Phi_0)(w) \cap (\nabla f(w) + N_\Omega(w)) \neq \emptyset\}$$

is finite, then the algorithm DCAC-MPCC applied with $\mu > \frac{L}{c\alpha}$ is well defined and converges in a finite number of iterations of p to a point $\overline{w} = (\overline{x}, \overline{y}, \overline{z}) \in \Omega$ which satisfies

$$\partial(-\mu\tilde{\Phi}_{\overline{w}})(\overline{w}) \subset \nabla f(\overline{w}) + N_\Omega(\overline{w}).$$

If moreover $\langle \overline{y}, \overline{z} \rangle = 0$ and MPCC-LICQ holds at $\overline{w}$, then $\overline{w}$ is a strongly stationary point for MPCC.

Proof. According to Item 4 in Theorem 6.5, for any $w^0 \in \Omega$, the whole sequence (w^k) generated by Step 1 in DCACM converges. Then for each iteration p, the element $\overline{w}^p$ is well defined. Therefore Step 2 is well defined.

We now prove that Step 3 is well defined. Let $w^* \in \mathrm{Ext}(\partial(-\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p))$ be such that $\overline{w}^p$ is not a solution of (6.9). We prove that $\varepsilon^{p+1} := f(\overline{w}^p) + \mu\Phi_0(\overline{w}^p) - f(\tilde{w}^p) - \mu\Phi_0(\tilde{w}^p)$ is positive where $\tilde{w}^p$ is a solution of (6.9). Since $\tilde{w}^p$ solves (6.9) and $\overline{w}^p$ does not solve it, we have:

$$f_1(\tilde{w}^p) - \langle \nabla f_2(\overline{w}^p) + \mu w^*, \tilde{w}^p \rangle < f_1(\overline{w}^p) - \langle \nabla f_2(\overline{w}^p) + \mu w^*, \overline{w}^p \rangle.$$

The above inequality can be written as follows:

$$f_1(\tilde{w}^p) - f_1(\overline{w}^p) - \langle \nabla f_2(\overline{w}^p) + \mu w^*, \tilde{w}^p - \overline{w}^p \rangle < 0. \tag{6.10}$$

Given that $\nabla f_2(\overline{w}^p) + \mu w^* \in \partial(f_2 - \mu\Phi_{\overline{w}^p})(\overline{w}^p)$ and $f_2 - \mu\tilde{\Phi}_{\overline{w}^p}$ is a convex function, we have:

$$\langle \nabla f_2(\overline{w}^p) + \mu w^*, \tilde{w}^p - \overline{w}^p \rangle \leq f_2(\tilde{w}^p) - \mu\tilde{\Phi}_{\overline{w}^p}(\tilde{w}^p) - f_2(\overline{w}^p) + \mu\tilde{\Phi}_{\overline{w}^p}(\overline{w}^p).$$

We apply the previous inequality in (6.10) and we obtain:

$$f_1(\tilde{w}^p) - f_1(\overline{w}^p) - \Big(f_2(\tilde{w}^p) - \mu\tilde{\Phi}_{\overline{w}^p}(\tilde{w}^p) - f_2(\overline{w}^p) + \mu\tilde{\Phi}_{\overline{w}^p}(\overline{w}^p)\Big) < 0$$

which implies

$$f(\tilde{w}^p) + \mu\tilde{\Phi}_{\overline{w}^p}(\tilde{w}^p) < f(\overline{w}^p) + \mu\tilde{\Phi}_{\overline{w}^p}(\overline{w}^p). \tag{6.11}$$

Since for any $a \geq 0$ and $b \geq 0$, we have $a + b - \sqrt{a^2+b^2} \leq \min\{a, b\}$, we have:

$$\forall w \in \Omega,\ \Phi_0(w) \leq \tilde{\Phi}_{\overline{w}^p}(w). \tag{6.12}$$

In another way, given that

$$\sum_{\substack{i=1 \\ \overline{y}_i^p + \overline{z}_i^p = 0}}^{m} \min\{\overline{y}_i^p, \overline{z}_i^p\} = \sum_{\substack{i=1 \\ \overline{y}_i^p + \overline{z}_i^p = 0}}^{m} \left(\overline{y}_i^p + \overline{z}_i^p - \sqrt{(\overline{y}_i^p)^2 + (\overline{z}_i^p)^2}\right) = 0$$

with $\overline{w}^p := (\overline{x}^p, \overline{y}^p, \overline{z}^p)$, we deduce that:

$$\Phi_0(\overline{w}^p) = \tilde{\Phi}_{\overline{w}^p}(\overline{w}^p). \tag{6.13}$$

Given that $\tilde{w}^p \in \Omega$, by (6.11), (6.12) and (6.13), we have:

$$f(\tilde{w}^p) + \mu\Phi_0(\tilde{w}^p) < f(\overline{w}^p) + \mu\Phi_0(\overline{w}^p).$$

Therefore, $\varepsilon^{p+1} > 0$. This finally proves that Step 3 is well defined, then the algorithm is well defined.

We now prove that the algorithm finishes in a finite number of iterations of p. According to Item 3 in Theorem 6.5, we have, for any iteration p:

$$\begin{aligned}
f(\overline{w}^{p+1}) + \mu\Phi_0(\overline{w}^{p+1}) \quad &\leq \quad f(w^0) + \mu\Phi_0(w^0) + M\sum_{k\geq 0}|\rho^{k+1} - \rho^k| \\
&< \quad f(\tilde{w}^p) + \mu\Phi_0(\tilde{w}^p) + M\frac{\varepsilon^{p+1}}{M} \text{ by construction of } (\rho^k)_k \\
&= \quad f(\tilde{w}^p) + \mu\Phi_0(\tilde{w}^p) + f(\overline{w}^p) + \mu\Phi_0(\overline{w}^p) - f(\tilde{w}^p) - \mu\Phi_0(\tilde{w}^p) \\
&= \quad f(\overline{w}^p) + \mu\Phi_0(\overline{w}^p).
\end{aligned}$$

This proves that for any iteration p, we have $\overline{w}^{p+1} \notin \{\overline{w}^0, \cdots, \overline{w}^p\}$. At the same time, according to Item 2 in Theorem 6.5, for any iteration p, we have

$$\overline{w}^p \in \{w \in \Omega \mid \partial(-\mu\Phi_0)(w) \cap (\nabla f(w) + N_\Omega(w)) \neq \emptyset\}$$

which is a finite set. Then the algorithm converges in a finite number of iterations of p.

Let p be the last iteration of the algorithm. Let $w^* \in \mathrm{Ext}(\partial(-\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p))$. Given that p is the last iteration, $\overline{w}^p$ solves (6.9), then the first order optimality condition leads to $\overline{w}^p + \nabla f(\overline{w}^p) - \mu w^* - \overline{w}^p \in -N_\Omega(\overline{w}^p)$, which implies that $\mu w^* \in \nabla f(\overline{w}^p) + N_\Omega(\overline{w}^p)$. That is true for all $w^* \in \mathrm{Ext}(\partial(-\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p)$, then

$$\mathrm{Ext}(\partial(-\mu\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p)) \subset \nabla f(\overline{w}^p) + N_\Omega(\overline{w}^p).$$

Since $\partial(-\mu\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p) = \mathrm{conv}\left(\mathrm{Ext}(\partial(-\mu\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p))\right)$ and $\nabla f(\overline{w}^p) + N_\Omega(\overline{w}^p)$ is a convex set, we deduce that:

$$\partial(-\mu\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p) \subset \nabla f(\overline{w}^p) + N_\Omega(\overline{w}^p).$$

We assume that $\langle \overline{y}^p, \overline{z}^p \rangle = 0$ and MPCC-LICQ holds that $\overline{w}^p$. The function $\tilde{\Phi}_{\overline{w}^p}$ clearly satisfies assumptions $\mathcal{H}_1$, $\mathcal{H}_3$ and $\mathcal{H}_4$ in Section 4. According to Inequality (6.12), the function $\tilde{\Phi}_{\overline{w}^p}$ satisfies assumption $\mathcal{H}_2$ with the same constant c as for Φ_0. We can observe that the function $\tilde{\Phi}_{\overline{w}^p}$ satisfies the assumption (5.2) at $\overline{w}^p$, then according to Theorem 5.4, $\overline{w}^p$ is a strongly stationary point for MPCC. $\square$

7 Numerical examples

To give more validation of our theoretical results, we test the algorithm on a set of MPCCs derived from MacMPEC collection [17]. The experiments were performed on Windows 10 Pro, with 3.20 GHz Intel using 5 cores and 8GB RAM. The DCAC-MPCC has been implemented in MATLAB (R2014a).

We only consider the case where g and h are affine functions. In Step 1, we consider the DC-descomposition $f(w) = f_1(w) - f_2(w)$ with $f_1(w) := 0.5\alpha\|w\|^2$ and $f_2(w) := 0.5\alpha\|w\|^2 - f(w)$ with $\alpha > 0$ large enough in order that f_2 is strongly convex on Ω. We solve (6.6) using quadprog on MATLAB. Step 1 finishes when $\|w_{k+1} - w_k\|_\infty < 10^{-1}$. When we consider a smaller tolerance, the number of iterations of p decreases but at the same time, the number of iterations in Step 1 increases, then the CPU time of the whole algorithm increases also.

In Step 2, we know that (6.9) has a unique solution. Given $\tilde{w}^p$ a solution of (6.9), we deduce from the uniqueness of solution of (6.9) that $\overline{w}^p$ is a solution of (6.9) if and only if $\overline{w}^p = \tilde{w}^p$. Then, for each $w^* \in \mathrm{Ext}(\partial(-\tilde{\Phi}_{\overline{w}^p})(\overline{w}^p))$, we test if $\|\overline{w}^p - \tilde{w}^p\|_\infty < 10^{-3}$, where $\tilde{w}^p$ is the solution (6.9). The program (6.9) is also solved by quadprog on MATLAB.

We obtained the following results. The real value is the optimal value of the optimization problem, the obtained value is the value that we obtained. The number of iterations corresponds to the number of iterations of p in DCAC-MPCC.

Name	Real value	Obtained value	Iterations	n	m	q	r
bilevel1	0	$-7.1054 \cdot 10^{-15}$	1	16	6	1	8
bilevel2	-6600	-6600	46	32	12	0	16
ex9.1.2	-6.25	-6.25	2	10	4	0	5
ex9.2.1	17	17	2	10	4	0	5
flp4.1	0	$3.6472 \cdot 10^{-4}$	1994	110	30	30	30
bard1	17	17	4	8	3	0	3
bard1m	17	17	4	8	3	0	3

8 Conclusion and future works

To our knowledge, this algorithm is the first DC Algorithm which allows for converging to a strongly stationary point for MPCC. In this paper we do not prove that this algorithm converges to a feasible point for MPCC, which is a weakness of this article and constitutes a possible extension of this work. In a future work a natural extension would be to consider the case where Ω is not convex, for example, with a linearization of the constraints. Moreover, numerical simmulations could be made. A new work (see e.g. [22]) proposes an acceleration of DCA. This acceleration could be applied to a DC algorithm for MPCCs.

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