J Optim Theory Appl (2014) 163:351–354
DOI 10.1007/s10957-013-0518-y

A Note on the Paper "Linear Complementarity Problems Over Symmetric Cones: Characterization of Qb-Transformations and Existence Results"

Julio López · Rúben López · Héctor Ramírez

Received: 7 December 2013 / Accepted: 27 December 2013 / Published online: 22 January 2014

Abstract In this note, we correct a mistake in the paper (López et al., J Optim Theory Appl 159(3):741–768, 2013).

Keywords Linear complementarity problem · Symmetric cone · García's transformation

Mathematical Subject Classification (2010) 90C33 · 17C55

1 Introduction

Recently, López et al. [1] introduced new classes of linear transformations on symmetric cones, in particular, the class of García's transformation. With this new class, they established coercive and noncoercive existence results for the symmetric cone linear complementarity problem (SCLCP). Unfortunately, the statement of Lemma 4.1, Parts (d), (e), and (f) in [1], which has been used to prove some results, is not true in general. In this note, we give the correct Proposition 2.1 and modify the formula-

J. López (✉)
Facultad de Ingeniería, Universidad Diego Portales, Ejército 441, Santiago, Chile
e-mail: julio.lopez@udp.cl

R. López
Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Alonso Ribera 2850, Concepción, Chile
e-mail: rlopez@ucsc.cl

H. Ramírez
Departamento de Ingeniería Matemática, Centro de ModelamientoMatemático (CNRS UMI 2807), FCFM, Universidad de Chile, Blanco Encalada 2120, Santiago, Chile
e-mail: hramirez@dim.uchile.cl

tions of the corresponding results. We also give a counterexample in the semidefinite programming (SDP) case to illustrate that Theorem 4.1 in [1] is not true for general Euclidean Jordan Algebras.

2 Errata Corrige

The following statement appears in Lemma 4.1 of [1].

Lemma 2.1 *Let $\{x^k\}$ be a sequence of solutions to* (ASCLCP$_k$)*:*

$$y^k := L(x^k) + q + \theta_k d \in \mathcal{K},\ \langle d, x^k\rangle \le \sigma_k, \langle y^k, x^k\rangle = 0 \text{ and } \theta_k(\sigma_k - \langle d, x^k\rangle) = 0.$$

such that $\langle d, x^k\rangle = \sigma_k$ for all $k \in \mathbb{N}$ and $\frac{x^k}{\sigma_k} \to v$ for some $v \in \mathcal{K}$. Then

(a) $v \in \mathrm{SOL}(L, \mathcal{K}, \tau_v d)$ *with* $\tau_v = -\langle L(v), v\rangle \ge 0$. *Moreover, there exists a nonempty subindex set $J_v \subseteq \{1, \dots, r\}$, a Jordan frame $\{e_1, \dots, e_r\}$ and a subsequence $\{k_m\}$ such that*

(b) $\{e_1^{k_m}, \dots, e_r^{k_m}\} \to \{e_1, \dots, e_r\}$ *and* $\lambda(\frac{x^{k_m}}{\sigma_{k_m}}) \to \lambda(v)$ *as* $m \to +\infty$*; thus* $y^{k_m} \to \gamma := (\lambda_1(v)\langle d, e_1\rangle, \dots, \lambda_r(v)\langle d, e_r\rangle) \in \Delta$.

(c) $y^{k_m} \in \mathrm{ri}(\Delta_{J_v})$*; i.e.,* $\mathrm{supp}\{\lambda(\frac{x^{k_m}}{\sigma_{k_m}})\} = J_v$ *and* $\lambda(y^{k_m})\Big|_{J_v} = 0$ *for all* $m \in \mathbb{N}$. *As a consequence, vectors $\lambda(y^{k_m})$ have at least $|J_v|$ zeros, which implies that* $\lambda_i^{\uparrow}(y^{k_m}) = 0$ *for all* $i = 1, \dots, |J_v|$*, and* $\mathrm{supp}\{\lambda(v)\} \subseteq J_v$. *Finally, for every* $z \in \mathcal{K} \setminus \{0\}$ *with* $\mathrm{supp}\{\lambda(z)\} \subseteq J_v$ *one has*

(d) $\langle y^{k_m}, z\rangle = 0$ *for all* $m \in \mathbb{N}$*;*

(e) $\left\langle L(x^{k_m}) + q, \frac{z}{\langle d,z\rangle}\right\rangle = \langle L(x^{k_m}) + q, v\rangle$ *for all* $m \in \mathbb{N}$*;*

(f) $\langle L(v), \frac{z}{\langle d,z\rangle}\rangle = \langle L(v), v\rangle$.

Parts (a), (b), and (c) above have been properly proved. However, the proof of Part (d) contains a mistake. Therein, one needs that $\lambda_i^{\uparrow}(z) = 0$ for all $i = 1, \dots, r - |J_v|$ be fulfilled instead of $\lambda_i^{\uparrow}(z) = 0$ for all $i = |J_v| + 1, \dots, r$, where $\lambda^{\uparrow}(z)$ denotes the vector of eigenvalues of z, whose components are arranged in the nondecreasing order. Consequently, items (e) and (f) are no longer true.

The latter leads to the following wrong sentence appearing in Proposition 4.2 of [1].

Proposition 2.1 *If $L \in G^{\#}$, then* $\mathrm{SOL}(L, \mathcal{K}, q)^{\infty} \subseteq \mathrm{SOL}(L, \mathcal{K}, 0) \cap \{-q\}^{+}$.

Indeed, the proof given in [1] correctly shows that $\mathrm{SOL}(L, \mathcal{K}, q)^{\infty} \subseteq \{-q\}^{+}$ provided that Parts (e) and (f) of Lemma 4.1 are true. Hence, since Parts (d), (e), and (f) are no longer true, the corrected version of Proposition 4.2 in [1] now reads as follows.

Proposition 2.2 *If $L \in G$, then* $\mathrm{SOL}(L, \mathcal{K}, q)^{\infty} \subseteq \mathrm{SOL}(L, \mathcal{K}, 0)$.

Finally, the proof of the following sentences, appearing in Theorem 4.1 of [1], also uses Lemma 4.1.

Theorem 2.1 *Let $q \in \mathbb{V}$ and $L \in G^{\#}$.*

(a) If $q \in \mathrm{SOL}(L, \mathcal{K}, 0)^{+}$, then $\mathrm{SOL}(L, \mathcal{K}, q)$ is nonempty (possibly unbounded);
(b) If $q \in \mathrm{int}[\mathrm{SOL}(L, \mathcal{K}, 0)^{+}]$, then $\mathrm{SOL}(L, \mathcal{K}, q)$ is nonempty and compact.

It is interesting to note that this result is indeed true when $\mathcal{K} = \mathbb{R}^n_+$ (cf. [[2], Theorems 9, 11]). However, this is not necessarily true for other Euclidean Jordan Algebras. This is shown via the next counterexample in the SDP case.

Example 2.1 Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The pair of primal and dual SDP problems

$$\min\{\langle C, X\rangle : \langle A, X\rangle \geq 2,\ X \in S^2_+\} \tag{1}$$

and

$$\max\{2t : C - tA \in S^2_+,\ t \geq 0\}, \tag{2}$$

do not have a primal-dual optimal solution. Indeed, it is easy to check that $t = 0$ is the only feasible solution of (2). If we denote by X_{ij} the (i, j)-th entry of X, it is easy as well to check that X is feasible for (1) iff

$$X_{12} = X_{21} \geq 1,\quad X_{11} \geq 0,\quad X_{22} \geq 0 \text{ and } X_{11}X_{22} \geq 1.$$

Hence, (1) is feasible but does not have an optimal solution (because its optimal value is 0 but it is never achieved).

Now, let $L : S^3 \to S^3$ be a linear transformation and $Q \in S^3$ be defined as follows:

$$L\begin{pmatrix} X & u \\ u^\top & t \end{pmatrix} := \begin{pmatrix} -tA & 0 \\ 0^\top & \langle A, X\rangle \end{pmatrix} \quad \text{and} \quad Q := \begin{pmatrix} C & 0 \\ 0^\top & -2 \end{pmatrix}.$$

It is easy to check that L is monotone (hence $L \in G^{\#}$). Moreover, $\mathrm{Feas}(L, S^n_+, Q) \neq \emptyset$ since

$$\begin{pmatrix} X & 0 \\ 0^\top & 0 \end{pmatrix} \in \mathrm{Feas}(L, S^n_+, Q) \quad \text{for} \quad X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The solution of the corresponding SDLCP satisfies

$$X \in S^2_+,\quad t \geq 0,\quad C - tA \in S^2_+,\quad \langle A, X\rangle - 2 \geq 0,\quad \langle X, C - tA\rangle = 0$$

and

$$t(\langle A, X\rangle - 2) = 0,$$

or equivalently, any solution of $\mathrm{SDLCP}(L, S^3_+, Q)$ yields an (X, t) one for the above pair of primal and dual SDPs. Since this pair do not have primal-dual solutions, we conclude that $\mathcal{S}(L, S^n_+, Q) = \emptyset$.

The above example shows that Theorem 2.1, Part (a) [and consequently Part (b)] is not true. Once again, the problem comes from the fact that its proof is based on Lemma 2.1, Part (e).

It is important to note that Corollary 4.1 in [1] is still correct. Let us recall this result.

Corollary 2.1 If $L \in G$, then $L \in R_0$ iff $L \in Q_b$.

Indeed, $L \in Q_b$ clearly implies that $L \in R_0$. In the opposite direction, it suffices to note that if $L \in R_0 \cap G$, then $L \in Q$ (see [3]).

Acknowledgments This research was supported by CONICYT-Chile, via FONDECYT Projects 11110188 (J. López), 1100919 (R. López), 1110888 (H. Ramírez). The third author was also supported by BASAL Project (Centro de Modelamiento Matemático, Universidad de Chile).

References

1. López, J., López, R., Ramírez, C.H.: Linear complementarity problems over symmetric cones: characterization of Qb-transformations and existence results. J. Optim. Theory Appl. **159**(3), 741–768 (2013). doi:10.1007/s10957-012-0116-4
2. Gowda, S.M., Pang, J.S.: The basic theorem of complementarity revisited. Math. Program. **58**, 161–177 (1993)
3. Karamardian, S.: An existence theorem for the complementarity problem. J. Optim. Theory Appl. **19**, 227–232 (1976)