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# Characterizing **Q**-linear transformations for semidefinite linear complementarity problems

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ABSTRACT

In this paper we introduce a new class, called **F**, of linear transformations defined from the space of real $n \times n$ symmetric matrices into itself. Within this new class, we show the equivalence between **Q**- and $\mathbf{Q}_b$-transformations. We also provide conditions under which a linear transformation belongs to **F**. Moreover, this class, when specialized to square matrices of size $n$, turns out to be the largest class of matrices for which such equivalence holds true in the context of standard linear complementary problems.

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## 1. Introduction

This paper is devoted to the study of the existence of solutions of linear complementarity problems over the cone $S^n_+$ of real $n \times n$ symmetric positive semidefinite matrices. The latter is usually called *semidefinite linear complementarity problem* (SDLCP). Recall that, given a linear transformation $L$, defined from the space of real $n \times n$ symmetric matrices $S^n$ into itself (for short $L \in \mathcal{L}(S^n)$), and a matrix $Q \in S^n$, the SDLCP consists in finding a matrix $\bar{X}$ such that

$$\bar{X} \in S^n_+, \qquad \bar{Y} = L(\bar{X}) + Q \in S^n_+ \quad \text{and} \quad \langle \bar{Y}, \bar{X} \rangle = 0, \tag{1}$$

where $\langle X, Y \rangle := \operatorname{tr}(XY) = \sum_{i,j=1}^n X_{ij} Y_{ij}$ denotes the trace of the (matrix) product $XY$. In the sequel, this problem will be denoted by $\text{SDLCP}(L, S^n_+, Q)$, and its solution will be denoted by $\text{SOL}(L, S^n_+, Q)$. Also, its feasible set is defined to be $\text{FEAS}(L, S^n_+, Q) := \{X \in S^n_+ : L(X) + Q \in S^n_+\}$.

The SDLCP was first introduced in a geometric form by Kojima et al. [1] and its applications include primal–dual semidefinite linear programs, control theory, linear and bilinear matrix inequalities, among others. This problem can be seen as a generalization of the (standard) *linear complementarity problem* LCP [2]. However, since the cone $S^n_+$ is nonpolyhedral, LCP theory cannot be trivially generalized to the SDLCP context. It is also a particular case of a cone complementarity problem, which turns out to be a particular case of a variational inequality problem [3]. Nevertheless, the direct application of existing results does not take advantage of its rich matrix structure. For more details, see [4,5] and the references therein.

On the other hand, when solving the $\text{LCP}(M, q)$ (for some given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$):

$$\text{find } \bar{x} \in \mathbb{R}^n_+ \quad \text{such that } \bar{y} = M\bar{x} + q \in \mathbb{R}^n_+ \quad \text{and} \quad \langle \bar{y}, \bar{x} \rangle = 0$$

by means of splitting methods, or when taking it as a basis for more sophisticated algorithms (see, for instance, [2, Chapter 5]), a specific class of matrices naturally emerges. This class, denoted by Q, contains all matrices $M \in \mathbb{R}^{n\times n}$ such that $\text{LCP}(M, q)$ has solutions for all $q$. This class also plays a relevant role in perturbation theory (see, for instance, [2,6]

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and the references therein). These motivations explain the important effort made in order to characterize the class Q. In particular, it is usual to analyze when the class Q coincides with the smaller class $Q_b$, where the latter consists of all matrices $M \in \mathbb{R}^{n\times n}$ such that the solution set of LCP$(M, q)$ is not empty and bounded for all $q$. In this context, Flores and López [6] introduce a new class of matrices (called $F_1$ therein) and prove that $Q = Q_b$ holds true within that class. This result generalizes previous ones of the same kind (e.g. [7, Theorem 1.2]).

Here, our aim is to extend the class $F_1$, and some of the results of [6], to the SDLCP framework. Actually, we define a large class of linear transformations, called **F**, for which it holds that $\mathbf{Q} = \mathbf{Q_b}$ (in the sense of linear transformations in $\mathcal{L}(S^n)$).[1,2] Then, we study its relations with LCPs, which motivates the definition of two subclasses; $\mathbf{F_1}$ and $\mathbf{F_2}$. This permits to show that class **F**, when specialized to matrices, is actually larger than $F_1$. Hence, as a by-product of our analysis, characterization $Q = Q_b$ is now proved in a larger class than $F_1$, which constitutes a novelty and an improvement of former results in LCP theory. Then, we provide conditions under which a linear transformation belongs to these subclasses. Finally, we illustrate these conditions with some well-known linear transformations such as Lyapunov functions $L_A(X) := A^\top X + XA^\top$, among others.

This paper is organized as follows. Section 2 is dedicated to the preliminaries. It is split into two subsections; first one recalls some basic results on matrix analysis, while second one summarizes some classes of linear transformations in $\mathcal{L}(S^n)$ with their respective connections. In Section 3, we established our main results described in the paragraph above. For this, in a first subsection, we recall known characterizations of classes $\mathbf{Q}$ and $\mathbf{Q_b}$ obtained via a recession analysis.

## 2. Preliminaries

In this section, we describe our preliminaries results. They are presented in two subsections; first one contains notations and some well-known matrix results needed in the sequel, while the second one recalls existing classes of linear transformations that are frequently used in the SDLCP theory.

### 2.1. Notation and basic results on matrix analysis

Some matrix operations are extensively used throughout this paper. For instance, we mention the trace and the diagonal of a square matrix $X = (X_{ij}) \in \mathbb{R}^{n\times n}$, defined by $\mathrm{tr}(X) := \sum_{i=1}^n X_{ii}$ and $\mathrm{diag}(X) := (X_{11}, X_{22}, \ldots, X_{nn})^\top$, respectively. The notion of a *submatrix* is also very useful in the sequel. For an $n \times n$ matrix $X = (X_{ij})$ and index sets $\alpha, \beta \subseteq \{1, \ldots, n\}$, we write $X_{\alpha\beta}$ to denote the submatrix of $X$ whose entries are $X_{ij}$ with $i \in \alpha$ and $j \in \beta$. When $\alpha = \beta$, $X_{\alpha\alpha}$ is usually called the *principal submatrix of X corresponding to* $\alpha$. In particular, when $\alpha = \{1, \ldots, k\}$ $(1 \le k \le n)$, $X_{\alpha\alpha}$ is called the *leading principal submatrix of X*.

Additionally, the Hadamard product has an important role in our approach. We recall that this operation is defined by $X * Y := (X_{ij}Y_{ij}) \in \mathbb{R}^{m\times n}$ for all $X = (X_{ij}),\ Y = (Y_{ij}) \in \mathbb{R}^{m\times n}$.

It is well-known that the set $S^n$ of real $n \times n$ symmetric matrices is a finite dimensional real Hilbert space when it is equipped with the inner product $\langle X, Y\rangle = \mathrm{tr}(XY)$. As usual, this product defines a (Frobenius) norm $\|X\|_F := \sqrt{\langle X, X\rangle} = \sqrt{\sum_{i=1}^n \lambda_i(X)^2}$, where $\lambda_i(X)$ stands for the $i$-th eigenvalue (arranged in nonincreasing order) of $X$. Thus, $\|X\|_F = \|\lambda(X)\|$ for all $X \in S^n$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and we have set $\lambda(X) := (\lambda_1(X), \ldots, \lambda_n(X))^\top$. Also, $0_n$ and $I_n$ denote the zero and the identity matrices, respectively, of size $n$, but index $n$ will be omitted if the size is clear from the context. Similarly, $\mathbb{1}_n$ (or $\mathbb{1}$) denotes the $n \times n$ matrix whose entries are all equal to 1.

Finally, for a vector $q \in \mathbb{R}^n$, we define $\mathrm{Diag}(q)$ as the diagonal matrix of size $n$ whose diagonal entries are given by the entries of $q$.

We end this subsection by recalling matrix properties that we shall employ throughout this paper. Their proofs and more details can be found in [8–10].

**Proposition 1.** *The following results hold:*

(a) *For any* $X \in \mathbb{R}^{n\times n}$ *and any orthogonal matrix* $U \in \mathbb{R}^{n\times n}$, *it holds that* $\mathrm{tr}(X) = \mathrm{tr}(X^\top) = \mathrm{tr}(UXU^\top)$. *Moreover, when X is a symmetric matrix with the following block structure* $X = \begin{pmatrix} A & B \\ B^\top & C\end{pmatrix} \in S^n_+$, *then it holds that* $\mathrm{tr}(A)\mathrm{tr}(C) \ge \mathrm{tr}(BB^\top)$;

(b) *(Von Neumann–Theobald's inequality) For any* $X, Y \in S^n_+$, *it holds that* $\langle X, Y\rangle \le \mathrm{diag}(X)^\top \mathrm{diag}(Y)$, *with equality if and only if X and Y are simultaneously diagonalizable (that is, there exists an orthogonal matrix U such that* $X = U\,\mathrm{Diag}(\lambda(X))U^\top$ *and* $Y = U\,\mathrm{Diag}(\lambda(Y))U^\top$*)*;

(c) *(Fejer's theorem) For any* $X \in S^n$, *it holds that* $\langle X, Y\rangle \ge 0$ *for all* $Y \in S^n_+$ *if and only if* $X \in S^n_+$. *Moreover,* $\langle X, Y\rangle > 0$ *for all* $Y \in S^n_+ \setminus \{0\}$ *if and only if* $X \in S^n_{++}$, *where* $S^n_{++}$ *denotes the cone of real* $n \times n$ *symmetric positive definite matrices;*

(d) *Let* $X, Y \in S^n_+$. *If* $\langle X, Y\rangle = 0$, *then X and Y commute (that is* $XY = YX$*)*;

[1] Recall that classes Q and $Q_b$ are word-by-word extended to the SDLCP framework as follows: *a linear transformation* $L \in \mathcal{L}(S^n)$ *is said to be a* **Q**-*transformation* ($\mathbf{Q}_b$*-transformation) if* $\mathrm{SOL}(L, S^n_+, Q) \ne \emptyset$ *(and bounded), for all* $Q \in S^n$. *For the sake of notation, we simply say that* $L \in \mathbf{Q}$ $(L \in \mathbf{Q}_b)$.

[2] In order to avoid misunderstandings, bold letters (such as **Q**) will denote classes of linear transformation in $\mathcal{L}(S^n)$, whereas roman-type letters (such as Q) will denote classes of matrices in $\mathbb{R}^{n\times n}$.

(e) *(Simultaneous diagonalization) Let $X, Y \in S^n_+$. If $X$ and $Y$ commute, then $X$ and $Y$ are simultaneously diagonalizable;*
(f) *As a direct corollary of* (d) *and* (e), *it follows that for any $X, Y \in S^n_+$ such that $\langle X, Y\rangle = 0$, $X$ and $Y$ are simultaneously diagonalizable;*
(g) *If $X, Y \in S^n_+$, then $X * Y \in S^n_+$.*

### 2.2. Linear transformation review

The literature on SDLCP (see [11,4,12–14]) has already extended, from the LCP theory, most of the well-known classes of matrices used in that context. We list these classes here below. Let $L \in \mathcal{L}(S^n)$, we say that:

- $L$ is an $\mathbf{R_0}$-transformation if $\mathrm{SOL}(L, S^n_+, 0) = \{0\}$.
- $L$ is copositive (resp. strictly copositive) if $\langle L(X), X\rangle \geq 0$ (resp. $> 0$) for all $X \in S^n_+$ (resp. for all $X \in S^n_+, X \neq 0$).
- $L$ is monotone (resp. strongly or strictly monotone) if $\langle L(X), X\rangle \geq 0$ (resp. $> 0$) for all $X \in S^n$ (resp. for all $X \in S^n, X \neq 0$).
- $L$ has the $\mathbf{P}$-property if $[XL(X) = L(X)X \in -S^n_+ \Rightarrow X = 0]$.
- $L$ has the $\mathbf{Q_0}$-property if $[\mathrm{FEAS}(L, S^n_+, Q) \neq \emptyset \Rightarrow \mathrm{SOL}(L, S^n_+, Q) \neq \emptyset]$.
- $L$ has the $\mathbf{S}$-property if there is $X \in S^n_+$ such that $L(X) \in S^n_{++}$, or equivalently, there is $X \in S^n_{++}$ such that $L(X) \in S^n_{++}$.
- $L$ is normal if $L$ commutes with $L^\top$.
- $L$ is a star-transformation if $[V \in \mathrm{SOL}(L, S^n_+, 0) \Rightarrow L^\top(V) \in -S^n_+]$.
- $L$ has the $\mathbf{Z}$-property if $[X, Y \in S^n_+,\ \langle X, Y\rangle = 0 \Rightarrow \langle L(X), Y\rangle \leq 0]$.

The next proposition establishes some links between the classes mentioned above.

**Proposition 2.** *Let $L \in \mathcal{L}(S^n)$ and $Q \in S^n$ be given. The following relations hold:*

(a) *$L$ is monotone $\Longrightarrow$ $L$ is copositive;*
(b) *$L$ is strongly monotone $\Longrightarrow L \in \mathbf{P} \Longrightarrow L \in \mathbf{R_0}$;*
(c) *$L \in \mathbf{S} \Longleftrightarrow \mathrm{FEAS}(L, S^n_+, Q) \neq \emptyset$ for all $Q \in S^n$;*
(d) *$\mathbf{Q} = \mathbf{Q_0} \cap \mathbf{S}$;*
(e) *Let $L \in \mathbf{Z}$. Then, $L \in \mathbf{Q} \Longleftrightarrow L \in \mathbf{S} \Longleftrightarrow \exists L^{-1}$ such that $L^{-1}(S^n_+) \subseteq S^n_+$ (equivalently, $L^{-1}(S^n_{++}) \subseteq S^n_{++}$).*

**Proof.** Statement (a) is direct from the definitions. Statement (b) is proven in [4]. The equality in (d) follows from (c). Finally, statement (e) has been proved in [14, Theorem 6]. So, only relation (c) need some adaptations of previous results. This is explained with more details here below.

(c): ($\Leftarrow$) Let $D \in S^n_{++}$. By hypothesis $\mathrm{FEAS}(L, S^n_+, -D) \neq \emptyset$, that is, there exists $X \in S^n_+$ such that $Y = L(X) - D \in S^n_+$. From this, we get $L(X) = Y + D \in S^n_{++}$. Hence $L \in \mathbf{S}$.

($\Rightarrow$) By hypothesis there is $X \in S^n_+$ such that $L(X) \in S^n_{++}$. Fix $Q \in S^n$. It is clear that for $t > 0$ large enough, the matrix $tL(X) + Q$ is symmetric positive definite. But, since $tL(X) = L(tX)$, the latter implies that $tX \in \mathrm{FEAS}(L, S^n_+, Q)$. The desired equivalence follows. □

Proposition 2 shows the rich relations existing among the different classes defined in $\mathcal{L}(S^n)$. However, at this stage of the analysis, it is worth to point out that these relations are not necessarily the same we find for matrices in the LCP theory. We illustrate this point here below through one enlightening example (see [14]).

**Example 1.** When we work with matrices, we have $\mathrm{Z} \subseteq \mathrm{Q}_0$ (see [2, Theorem 3.11.6]). However, this inclusion is no longer true when we deal with linear transformation in $\mathcal{L}(S^n)$. Indeed, consider

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}.$$

So, for the Lyapunov function $L_A = A^\top X + XA^\top$, we have $L_A \in \mathbf{Z}$, $\mathrm{FEAS}(L_A, S^2_+, Q) \neq \emptyset$, and $\mathrm{SOL}(L_A, S^2_+, Q) = \emptyset$. Hence, $L_A \notin \mathbf{Q_0}$.

## 3. Characterizations of Q- and $\mathbf{Q_b}$-transformations

This section is devoted to the characterization of classes $\mathbf{Q}$ and $\mathbf{Q_b}$. In particular, we are interested in studying classes of functions in $\mathcal{L}(S^n)$ for which $\mathbf{Q}$ behaves similarly as $\mathbf{Q_b}$.

### 3.1. Known results based on recession analysis

In finite dimensional spaces, the notion of the asymptotic cone of a set becomes a fundamental tool in order to characterize its boundedness. For a nonempty set $A \subseteq \mathbb{R}^n$, this notion is defined as follows (e.g. [15]):

$$A^\infty := \left\{ v \in \mathbb{R}^n : \exists t_k \to +\infty\, \exists \{x^k\} \subseteq A \text{ such that } \frac{x^k}{t_k} \to v \right\}$$

(by convention, we set $\emptyset^\infty = \{0\}$). Indeed, it is well known that $A$ is bounded if and only if $A^\infty = \{0\}$ (e.g. [15, Proposition 2.1.2]). So, the notion of asymptotic cones arises naturally when we deal with the class $\mathbf{Q_b}$. The following technical lemma illustrates this point. Its proof can be found in [16, Proposition 2.5.6].

**Lemma 3.** *Let $L \in \mathcal{L}(S^n)$ be given.*

(a) $\bigcup_{Q \in S^n} \mathrm{SOL}(L, S^n_+, Q)^\infty = \mathrm{SOL}(L, S^n_+, 0)$;
(b) *If $L \in \mathbf{R_0}$, then $\mathrm{SOL}(L, S^n_+, Q)$ is bounded (possibly empty) for all $Q \in S^n$;*
(c) *$L \in \mathbf{R_0}$ if and only if there exists a constant $c > 0$ such that*

$$\|X\|_F \le c\|Q\|_F, \quad \text{for all } Q \in S^n \text{ and } X \in \mathrm{SOL}(L, S^n_+, Q).$$

As a direct consequence of Lemma 3 above, we establish the equivalence between the classes $\mathbf{Q}$ and $\mathbf{Q_b}$ within the class $\mathbf{R_0}$.

**Corollary 4.** *Let $L \in \mathbf{R_0}$. Then, $L \in \mathbf{Q_b} \Longleftrightarrow L \in \mathbf{Q}$.*

Our main goal is to prove the previous equivalence in a larger class of linear transformations in $\mathcal{L}(S^n)$. To do this, we need the following lemma that follows from the fact that $\mathcal{S}(L, S^n_+, 0)$ is a cone.

**Lemma 5.** *It holds that $\mathbf{Q_b} \subseteq \mathbf{R_0}$. Consequently, $\mathbf{Q_b} = \mathbf{Q} \cap \mathbf{R_0}$.*

### 3.2. The class of F-transformations and its subclasses

In the LCP context, Flores and López [6] introduce the following class of matrices.

**Definition 6.** A matrix $M \in \mathbb{R}^{n\times n}$ is said to be an $\mathrm{F}_1$-matrix if, for every $v \in \mathrm{SOL}(M, 0) \setminus \{0\}$, there exists a nonnegative diagonal matrix $\Gamma$ such that $\Gamma v \ne 0$ and $M^\top \Gamma v \in -\mathbb{R}^n_+$. Here $\mathrm{SOL}(M, q)$ denotes, for given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, the solution of problem $\mathrm{LCP}(M, q)$.

So, in [6], the equivalence of Corollary 4 is proven within the class $\mathrm{F}_1$. This class turns out to be larger than $R_0$, which makes the result interesting to be extended to our SDLCP framework. Inspired by that definition, we introduce the next new class of transformations in $\mathcal{L}(S^n)$.

**Definition 7.** We say that $L \in \mathcal{L}(S^n)$ is an $\mathbf{F}$-transformation or $L \in \mathbf{F}$, if for each $V \in \mathrm{SOL}(L, S^n_+, 0) \setminus \{0\}$ there exists a matrix $\mathcal{T}(V)$ such that

$$\text{(i) } \mathcal{T}(V) \in S^n_+, \qquad \text{(ii) } \langle \mathcal{T}(V), V\rangle > 0, \qquad \text{(iii) } L^\top(\mathcal{T}(V)) \in -S^n_+. \tag{2}$$

We now establish the main properties of the class $\mathbf{F}$. In particular, assertion (b) below extends Corollary 4 to this larger class.

**Theorem 8.** *Let $L \in \mathcal{L}(S^n)$ be given.*

(a) *If $L \in \mathbf{F} \cap \mathbf{S}$, then $L \in \mathbf{R_0}$;*
(b) *Let $L \in \mathbf{F}$. Then, $L \in \mathbf{Q_b} \Longleftrightarrow L \in \mathbf{Q}$.*

**Proof.** (a) Let $L \in \mathbf{F} \cap \mathbf{S}$. We argue by contradiction. Suppose that $L \notin \mathbf{R_0}$, that is, there exist $V \in \mathrm{SOL}(L, S^n_+, 0) \setminus \{0\}$. Since $L \in \mathbf{F}$, there exists a matrix $\mathcal{T}(V)$ satisfying (i)–(iii) in Definition 7. This together with Fejer's theorem (Proposition 1, Part (c)) implies that $\langle L(X) - V, \mathcal{T}(V)\rangle < 0$ for all $X \in S^n_+$. Consequently, $L(X) - V \notin S^n_+$ for all $X \in S^n_+$. Therefore, by Proposition 2, Part (c), it follows that $L \notin \mathbf{S}$, obtaining a contradiction.

(b) Obviously $L \in \mathbf{Q_b}$ implies $L \in \mathbf{Q}$. If $L \in \mathbf{Q}$, then $L \in \mathbf{S}$ (because Proposition 2, Part (d)). Thus, $L \in \mathbf{F} \cap \mathbf{S}$. By item (a) above we conclude that $L \in \mathbf{R_0}$, and consequently $L \in \mathbf{Q} \cap \mathbf{R_0}$. We thus conclude that $L \in \mathbf{Q_b}$ thanks to equality $\mathbf{Q_b} = \mathbf{Q} \cap \mathbf{R_0}$ established in Lemma 5. □

Next example, adapted from [6], shows that the inclusion established in Part (a) of Theorem 8 above is strict.

**Example 2.** Consider the matrix $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. For $\mathcal{M}$ given by (4), we have $\mathrm{SOL}(\mathcal{M}, S^2_+, 0) = \{0\}$. Thus, $\mathcal{M} \in \mathbf{R_0}$. However, it is easy to see that $\mathrm{FEAS}(\mathcal{M}, S^n_+, M) = \emptyset$. This together with Proposition 2, Part (c), implies that $\mathcal{M} \notin \mathbf{S}$.

To check whenever a linear transformation $L$ belongs to $\mathbf{F}$ can be a difficult task. This is mainly because there is no clear guide about how to chose, for a given $V \in \mathrm{SOL}(L, S^n_+, 0) \setminus \{0\}$, a matrix $\mathcal{T}(V)$ satisfying conditions (i)–(iii) in Definition 7. For this, we focus now on the subclass of $\mathbf{F}$ for which $\mathcal{T}(V)$ is chosen via a linear transformation of the form

$$\mathcal{T}(V) = \Lambda * V \tag{3}$$

for some $\Lambda \in S^n_+$. Recall that symbol $*$ denotes the Hadamard product defined in Section 2.1. From now on, this subclass of $\mathbf{F}$ will be denoted by $\mathbf{F_1}$.

**Remark 1.** Notice that, thanks to Part (g) of Proposition 1, condition (i) in Definition 7 becomes superfluous when $L \in \mathbf{F_1}$.

**Remark 2.** By the nature of its definition, the class **F** can be word-by-word extended to LCPs over closed convex cones (see [17]). Recall that this problem consists in replacing $S^n$ and $S^n_+$ by a finite dimensional real Hilbert space $\mathcal{H}$ and a closed convex cone $\mathcal{K}$, respectively, in the definition of SDLCP. Similarly, class $\mathbf{F_1}$ can be extended to the LCPs over symmetric cones (see [18,13]). In this case, $\mathcal{H} = \mathcal{V}$ is a Euclidean Jordan Algebra $(\mathcal{V}, \circ, \langle\cdot,\cdot\rangle)$, $\mathcal{K} = \{x \circ x : x \in \mathcal{V}\}$ is the cone of squares in $\mathcal{V}$ (see [19,18]), and we should employ a Schur-type product, suitable defined in [20] for this framework, instead of the Hadamard product $*$ in equality (3). This subject is beyond the scope of this article and we expect to approach it in future research.

The name $\mathbf{F_1}$ is justified by the close relation existing between this subclass and the original class $\mathrm{F}_1$ defined in the LCP framework (see Definition 6). Indeed, it is easy to verify that

$$\bar{X} \in \mathrm{SOL}(\mathcal{M}, S^n_+, \mathrm{Diag}(q)) \quad \Longrightarrow \quad \bar{x} := \mathrm{diag}(\bar{X}) \in \mathrm{SOL}(M, q)$$

$$\bar{x} \in \mathrm{SOL}(M, q) \quad \Longrightarrow \quad \bar{X} := \mathrm{Diag}(\bar{x}) \in \mathrm{SOL}(\mathcal{M}, S^n_+, \mathrm{Diag}(q)),$$

where the linear transformation $\mathcal{M} : S^n \to S^n$ is defined by

$$\mathcal{M}(X) := \mathrm{Diag}(M\mathrm{diag}(X)). \tag{4}$$

See, for instance, [21]. Thus, the mentioned relation is stated in the next proposition.

**Proposition 9.** *Let $M \in \mathbb{R}^n$. If $\mathcal{M}$ is given by (4), then*

$$M \in \mathrm{F}_1 \Longleftrightarrow \mathcal{M} \in \mathbf{F_1}.$$

**Proof.** We first point out that the transpose of $\mathcal{M}$ is given by $\mathcal{M}^\top(X) = \mathrm{Diag}(M^\top \mathrm{diag}(X))$.

($\Rightarrow$) Let $M \in \mathrm{F}_1$. If $V \in \mathrm{SOL}(\mathcal{M}, S^n_+, 0) \setminus \{0\}$, then $v = \mathrm{diag}(V) \in \mathrm{SOL}(M, 0)$. Clearly $v \neq 0$ (otherwise, since $V \in S^n_+$, $V$ should be null). So, by hypothesis there exists a nonnegative diagonal matrix $\Gamma$ such that $\Gamma v \neq 0$ and $M^\top \Gamma v \in -\mathbb{R}^n_+$. Thus, conditions (i)–(iii) of Definition 7 can be easily verified provided that $\Lambda = \Gamma \in S^n_+$. We then obtain that $\mathcal{M} \in \mathbf{F_1}$.

($\Leftarrow$) Let $\mathcal{M} \in \mathbf{F_1}$. If $v \in \mathrm{SOL}(M, 0) \setminus \{0\}$, then $V = \mathrm{Diag}(v) \in \mathrm{SOL}(\mathcal{M}, S^n_+, 0)$, and obviously $V \neq 0$. By hypothesis there exists a matrix $\Lambda \in S^n_+$ such that the linear transformation $\mathcal{T}$, given by (3), satisfies conditions (i)–(iii) in Definition 7. Take $\Gamma := \mathrm{Diag}(\mathrm{diag}(\Lambda))$. Clearly $\Gamma$ is a diagonal matrix with nonnegative entries. Moreover, since $\Gamma v = \mathrm{diag}(\Lambda * V) \neq 0$ and $M^\top \Gamma v = \mathrm{diag}(\mathcal{M}^\top(\Lambda * V)) \in -\mathbb{R}^n_+$, it follows that $M \in \mathrm{F}_1$. □

As a consequence of the analysis above, we realize that we have also extended the class of matrices for which the equivalence between Q and $\mathrm{Q_b}$ (in the LCP framework) holds true. Indeed, former result in [6] only deals with the class $\mathrm{F}_1$, which is smaller than the class **F** restricted to linear transformations $\mathcal{M}$ of the form (4). In other words, it is clear that $\mathcal{M} \in \mathbf{F}$ allows not only matrices $M \in \mathrm{F}_1$ (for instance, it allows one to choose, in Definition 6, a matrix $\Gamma$ which is not necessarily diagonal, provided that condition $\Gamma v \neq 0$ be replaced by the equivalent condition $v^\top \Gamma v > 0$; see example below). Thus, the equivalence established in Theorem 8 constitutes also an improvement of the existing one in LCP theory.

**Example 3.** In this example, we show that $\mathbf{F_1}$ is properly contained in **F**. Set

$$M = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is not difficult to see that $M \notin \mathrm{F}_1$. Consequently, by Proposition 9, $\mathcal{M} \notin \mathbf{F_1}$. However, the matrix

$$\mathcal{T}(V) := \begin{pmatrix} z & 0 \\ 0 & \frac{z}{2} \end{pmatrix}, \quad \text{when } V = \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} \in \mathrm{SOL}(\mathcal{M}, S^2_+, 0) \setminus \{0\},$$

satisfies conditions (i)–(iii) in Definition 7. Therefore, $\mathcal{M} \in \mathbf{F}$.

Another different way to check whenever a linear transformation belongs to **F** is via the study of its block structure. The next proposition establishes a criterium based on this information.

**Proposition 10.** *Let $L \in \mathcal{L}(S^n)$. Suppose that for any orthogonal matrix $U \in \mathbb{R}^{n\times n}$ and for any index set $\alpha = \{1, \ldots, k\} (1 \leq k \leq n)$, the existence of a solution $X \in S^n$ to the system*

$$X_{\alpha\alpha} \in S^{|\alpha|}_{++}, \qquad X_{ij} = 0, \quad \forall i, j \notin \alpha, \qquad [\widehat{L}_U(X)]_{\alpha\alpha} = 0, \qquad [\widehat{L}_U(X)]_{\alpha\bar{\alpha}} = 0, \qquad [\widehat{L}_U(X)]_{\bar{\alpha}\bar{\alpha}} \in S^{|\bar{\alpha}|}_{+}, \tag{5}$$

*where $\widehat{L}_U(X) := U^\top L(UXU^\top)U$ and $\bar{\alpha} = \{1, \ldots, n\} \setminus \alpha$, implies that there is a nonzero matrix $Y \in S^n$ satisfying*

$$Y_{\alpha\alpha} \in S^{|\alpha|}_{+}, \qquad Y_{ij} = 0, \quad \forall i, j \notin \alpha, \qquad [\widehat{L^\top_U}(Y)]_{\alpha\alpha} = 0, \qquad [\widehat{L^\top_U}(Y)]_{\alpha\bar{\alpha}} = 0, \qquad [\widehat{L^\top_U}(Y)]_{\bar{\alpha}\bar{\alpha}} \in -S^{|\bar{\alpha}|}_{+}. \tag{6}$$

*Then, L is an* **F***-transformation.*

**Proof.** Let $V$ be a nonzero solution of SDLCP$(L, S^n_+, 0)$. Consider an orthonormal matrix $U \in \mathbb{R}^{n\times n}$ whose columns are eigenvectors of $V$. It follows that

$$U^\top VU = \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix},$$

for some $Z \in S^k_{++}$ (actually $Z$ is a diagonal matrix containing all positive eigenvalues of $V$) and $k \in \{1, \dots, n\}$. Set $\alpha := \{1, \dots, k\}$. We proceed to show that $X := U^\top VU$ is a solution of (5). First, $X$ clearly satisfies the first two conditions of (5). Also, $\widehat{L}_U(X) = U^\top L(V)U$. So, since $L(V) \in S^n_+$, it follows that $[\widehat{L}_U(X)]_{\bar\alpha\bar\alpha} \in S^{|\bar\alpha|}_+$. Moreover, condition $\langle L(V), V\rangle = 0$ implies that the columns of $U$ can be chosen in order to be also a basis of orthonormal eigenvectors of $L(V)$ (cf. Proposition 1, Part (f)). This yields, on the one hand, to $[\widehat{L}_U(X)]_{\alpha\bar\alpha} = 0$ (because $\widehat{L}_U(X) = U^\top L(V)U$ is actually a diagonal matrix), and, on the other hand, to $[\widehat{L}_U(X)]_{\alpha\alpha} = 0$ (because of $\langle L(V), V\rangle = \langle \widehat{L}_U(X), X\rangle$). Hence, there exists a nonzero solution $Y$ of (6).

We claim that matrix $\mathcal{T}(V) := UYU^\top$ satisfies conditions (i)–(iii) in (2). Indeed, it is obvious that $\mathcal{T}(V) \in S^n_+$. So, due to positive definiteness of $Z$ and Fejer's theorem (see Proposition 1, Part (c)), we obtain that $\langle \mathcal{T}(V), V\rangle = \langle Y_{\alpha\alpha}, Z\rangle > 0$. Finally, since $\widehat{L^\top_U}(Y) = U^\top L^\top(\mathcal{T}(V))U$ and $\widehat{L^\top_U}(Y) \in -S^n_+$ (consequence of (6)), it follows that $L^\top(\mathcal{T}(V)) \in -S^n_+$. We have thus deduced that $L$ is an **F**-transformation. □

**Remark 3.** In the implication stated in Proposition 10 above, we can choose an index set $\alpha \subseteq \{1, \dots, n\}$ not necessarily of the form $\{1, \dots, k\}$ for some $k \in \{1, \dots, n\}$. Indeed, given an orthonormal matrix $U \in \mathbb{R}^{n\times n}$ and an arbitrary index set $\alpha \subseteq \{1, \dots, n\}$, let $\tilde X$ be a matrix satisfying (5). We write $\alpha_i$ to denote the $i$-th component of $\alpha$. So, we define $P \in \mathbb{R}^{n\times n}$ as the permutation matrix such that the position $\alpha_i$ is switched with position $i$, for all $i \in \{1, \dots, |\alpha|\}$ (that is, if $\tilde x = Px$, then $x_i = x_{\alpha_i}$). Set $k := |\alpha|$. Since any permutation matrix is orthonormal, it follows that $\tilde U := UP^\top$ is orthonormal. Then, it is easy to note that $X := P\tilde XP^\top$ satisfies (5) when $\alpha$ is replaced by $\{1, \dots, k\}$ and $U$ is replaced by $\tilde U$. Thus, if the implication stated in Proposition 10 holds, we obtain the existence of $Y \in S^n$ solution of (6) for the same data (i.e. $\{1, \dots, k\}$ and $\tilde U$). Finally, it suffices to note $\tilde Y := P^\top YP$ satisfies (6) for the original $\alpha$ and $U$.

From now on, the class of transformations $L$ such that the implication stated in Proposition 10 holds true will be denoted by $\mathbf{F_2}$. Clearly, Proposition 10 above shows that $\mathbf{F_2}$ is a subclass of $\mathbf{F}$.

Once again, this subclass is closely related to the class $\mathrm{F}_1$, defined in the LCP framework (see Definition 6). Indeed, it is easy to see that a matrix $M \in \mathbb{R}^{n\times n}$ is an $\mathrm{F}_1$-matrix if and only if, for any nonempty set $\alpha \subseteq \{1, \dots, n\}$, the existence of a vector $x_\alpha \in \mathbb{R}^{|\alpha|}$ satisfying

$$x_\alpha > 0, \qquad M_{\alpha\alpha}x_\alpha = 0 \quad \text{and} \quad M_{\bar\alpha\alpha}x_\alpha \geq 0, \tag{7}$$

implies that there exists a nonzero vector $w_\alpha \in \mathbb{R}^{|\alpha|}_+$ such that

$$w_\alpha^\top M_{\alpha\alpha} = 0 \quad \text{and} \quad w_\alpha^\top M_{\alpha\bar\alpha} \leq 0. \tag{8}$$

So, keeping this characterization in mind, we establish the mentioned relation here below.

**Proposition 11.** *Let $M \in \mathbb{R}^n$ and consider $\mathcal{M}$ defined in (4). If $\mathcal{M} \in \mathbf{F_2}$, then $M \in \mathrm{F}_1$.*

**Proof.** Thanks to Remark 3, we can consider any arbitrary index set $\alpha \subseteq \{1, \dots, n\}$ in the definition of $\mathbf{F_2}$. Then, it suffices to note that systems (5) and (6) coincide with (7) and (8), respectively, when we consider $U = I$ (the identity matrix), $X_{\alpha\alpha} = \mathrm{Diag}(x_\alpha)$, $X_{ij} = 0$ for all $i, j \notin \alpha$, and $Y_{\alpha\alpha} = \mathrm{Diag}(w_\alpha)$, $Y_{ij} = 0$ for all $i, j \notin \alpha$. □

In the following proposition we list various classes of linear transformations that are contained in the classes $\mathbf{F_1}$ and $\mathbf{F_2}$.

**Proposition 12.** *$L \in \mathbf{F_1} \cap \mathbf{F_2}$ if any of the following conditions is satisfied:*

(a) *$L$ is a star-transformation;*
(b) *$L \in \mathbf{Z}$ and*
  (i) *$-L$ is copositive or*
  (ii) *$L$ is normal;*
(c) *$L \in \mathbf{R_0}$.*

**Proof.** In order to prove items (a) and (b), we split the proof into two parts; in the first one we prove that $L \in \mathbf{F_1}$ while in the second one we show that $L \in \mathbf{F_2}$. For both classes, item (c) is trivially verified by vacuity.

$L \in \mathbf{F_1}$:

(a) Let $V \in \mathrm{SOL}(L, S^n_+, 0) \setminus \{0\}$. Since $L$ is a star-transformation, we have $L^\top(V) \in -S^n_+$. Then, conditions (i)–(iii) of Definition 7 can be easily checked provided that $\mathcal{T}$ is of the form (3) with $\Lambda = \mathbb{1}$ (note that $\mathbb{1} \in S^n_+$). The result follows.

(b) Let $V \in \mathrm{SOL}(L, S^n_+, 0) \setminus \{0\}$, that is, $V, L(V) \in S^n_+$ and $\langle L(V), V\rangle = 0$. Since $L \in \mathbf{Z}$, we get $\langle L(V), L(V)\rangle \leq 0$, and consequently $L(V) = 0$. We proceed to prove both cases.

(i) If $-L$ is copositive, then $\langle L(tX + V), tX + V\rangle \leq 0$ for all $X \in S^n_+$ and for all $t > 0$. From this, after dividing by $t$ we get $t\langle L(X), X\rangle + \langle L(X), V\rangle \leq 0$ for all $t > 0$. Taking limit $t \searrow 0$ we obtain $\langle X, L^\top(V)\rangle \leq 0$ for all $X \in S^n_+$. From Fejer's theorem (Proposition 1, Part (c)), we conclude that $L^\top(V) \in -S^n_+$, that is, $L$ is a star-transformation. The desired result follows from (a).

(ii) Since $L$ is normal and $L(V) = 0$, we obtain that

$$\|L^\top(V)\|_F^2 = \langle V, L(L^\top(V))\rangle = \langle V, L^\top(L(V))\rangle = 0.$$

That is, $L^\top(V) = 0$, which is in particular a matrix in $-S^n_+$. Thus, the desired result follows again from (a).

$L \in \mathbf{F_2}$:

(a) Let $U$ be an orthonormal matrix of size $n$, $\alpha = \{1, \ldots, k\}$ $(1 \le k \le n)$ be a nonempty index set, and $X \in S^n$ a solution of system (5). It is easy to show that

$$V = UXU^\top = U \begin{pmatrix} X_{\alpha\alpha} & 0 \\ 0 & 0 \end{pmatrix} U^\top$$

is a nonzero solution of SDLCP$(L, S^n_+, 0)$. Indeed, $V = UXU^\top$, $L(V) = U\widehat{L}_U(X)U^\top \in S^n_+$ (because $X, \widehat{L}_U(X) \in S^n_+$) and $\langle L(V), V\rangle = \langle \widehat{L}_U(X), X\rangle = 0$. Since $L$ is a star-transformation, we have that $L^\top(V) \in -S^n_+$. Consequently, $\widehat{L}_U^\top(X) = U^\top L^\top(V)U \in -S^n_+$. On the other hand, since $X_{\alpha\alpha} \in S^{|\alpha|}_{++}$ and $[\widehat{L}_U^\top(X)]_{\alpha\alpha} \in -S^{|\alpha|}_+$ and the equality

$$\langle [\widehat{L}_U^\top(X)]_{\alpha\alpha}, X_{\alpha\alpha}\rangle = \langle \widehat{L}_U^\top(X), X\rangle = \langle X, \widehat{L}_U(X)\rangle = 0,$$

it follows that $[\widehat{L}_U^\top(X)]_{\alpha\alpha} = 0$. This together with condition $-\widehat{L}_U^\top(X) \in S^n_+$ implies that $[\widehat{L}_U^\top(X)]_{\alpha\bar{\alpha}} = [\widehat{L}_U^\top(X)]_{\bar{\alpha}\alpha} = 0$ (because of Proposition 1, Part (a)). Hence, $Y := X$ solves (6). We thus conclude that $L \in \mathbf{F_2}$

(b) Let $U$ be an orthonormal matrix of size $n$, $\alpha = \{1, \ldots, k\}$ $(1 \le k \le n)$ be a nonempty index set, and $X \in S^n$ a solution of system (5). As before, $V = UXU^\top$ is a nonzero solution of SDLCP$(L, S^n_+, 0)$. Since $L \in \mathbf{Z}$, we get $\langle L(V), L(V)\rangle \le 0$ and consequently $L(V) = 0$. Hence, $\widehat{L}_U(X) = U^\top L(V)U = 0$. Moreover, $-\widehat{L}_U$ is copositive when $-L$ is copositive and $\widehat{L}_U$ is normal when $L$ is normal. Thus, the arguments given in order to prove that $L \in \mathbf{F_1}$, but applied to $\widehat{L}_U$ instead of $L$, imply that $L \in \mathbf{F_2}$. □

**Remark 4.** The classes of linear transformations treated in Proposition 12 play an important role in the LCP theory. Indeed, star-transformations are usually studied in order to analyze the existence of LCPs (see [2] for usual LCPs and [22] for LCPs over symmetric cones). And, concerning **Z**-transformation in the context of SDLCPs, it is known that a linear transformation $L \in \mathcal{L}(S^n_+)$ has both the **Z** and **Q** properties if and only if the dynamical system $\frac{\mathrm{d}x}{\mathrm{d}t} + L(x) = 0$ is globally asymptotically stable and viable with respect to $S^n_+$ (see [14]).

### *3.3. Examples*

Some linear transformations in $\mathcal{L}(S^n)$ arise naturally in matrix theory and its applications. This is the case of Lyapunov, multiplicative and Stein transformations, that are defined, for a given $A \in \mathbb{R}^{n\times n}$, as follows:

- $L_A(X) = AX + XA^\top$, • $M_A(X) = AXA^\top$, • $S_A(X) = X - AXA^\top$.

We recall some properties of these transformations. Their proofs can be found in [12,14].

**Proposition 13.** *Let $A \in \mathbb{R}^{n\times n}$ be given.*

(a) $L_A^\top = L_{A^\top}$, $M_A^\top = M_{A^\top}$, *and* $S_A^\top = S_{A^\top}$*;*
(b) *If $A$ is normal (i.e. $AA^\top = A^\top A$) if and only if $L_A$, $M_A$ and $S_A$ are normal;*
(c) $L_A$, $S_A \in \mathbf{Z}$*.*

In the next proposition, we provide conditions on $A$ that ensure previous transformations belong to the classes $\mathbf{F_1}$ and $\mathbf{F_2}$.

**Proposition 14.** *Let $A \in \mathbb{R}^{n\times n}$ be given.*

(a) *If $A$ is normal, then $L_A, S_A \in \mathbf{F_1} \cap \mathbf{F_2}$;*
(b) *If $A$ is positive definite (i.e. $\langle Ax, x\rangle > 0$ for all nonzero $x \in \mathbb{R}^n$) or positive stable (i.e. all the eigenvalues of $A$ have a positive real part), then $L_A \in \mathbf{F_1} \cap \mathbf{F_2}$;*
(c) *If $A$ is positive definite or negative definite, then $M_A \in \mathbf{F_1} \cap \mathbf{F_2}$;*
(d) *If $A$ is Schur stable (i.e. all the eigenvalues of $A$ lie in the open unit disk), then $S_A \in \mathbf{F_1} \cap \mathbf{F_2}$.*

**Proof.** (a) By Proposition 13 we have that $L_A$ and $S_A$ are normal and **Z**-transformations. The result follows from Proposition 12, Part (b).

(b) Let $A$ be positive definite. Since [5, Theorem 5], the latter is equivalent to saying that $L_A$ is strongly monotone, which in turn by Proposition 2, Part (b), implies that $L_A \in \mathbf{R_0}$. Let $A$ be positive stable. Thanks to [4, Theorem 5], this implies that $L_A \in \mathbf{R_0}$. In both cases, the result follows from Proposition 12, Part (c).

(c) By [21, Theorem 3.3.2], the hypothesis is equivalent to saying that $M_A \in \mathbf{R_0}$. The result follows as in part (b).

(d) It is known from [23, Theorem 11] that the hypothesis is equivalent to $S_A \in \mathbf{P}$. Thus, Proposition 2, Part (b), implies that $S_A \in \mathbf{R_0}$. The result also follows as in part (b). □

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## References

[1] M. Kojima, S. Shindoh, S. Hara, Interior-point methods for the monotone semidefinite linear complementarity problems, SIAM J. Optim. 7 (1997) 86–125.
[2] R.W. Cottle, J.S. Pang, R.E. Stone, The Linear Complementarity Problem, SIAM, Philadelphia, 2009.
[3] S. Karamardian, An existence theorem for the complementarity problem, J. Optim. Theory Appl. 19 (1976) 227–232.
[4] M.S. Gowda, Y. Song, On semidefinite linear complementarity problems, Math. Program. 88 (2000) 575–587.
[5] T. Parthasaraty, D. Sampangi Raman, B. Sriparna, Relationship between strong monotonicity property, $P_2$-property, and the GUS-property in semidefinite linear complementarity problems, Math. Oper. Res. 27 (2002) 326–331.
[6] F. Flores-Bazán, R. López, Characterizing $Q$-matrices beyond $L$-matrices, J. Optim. Theory Appl. 127 (2005) 447–457.
[7] J.S. Pang, On $Q$-matrices, Math. Program. 17 (1979) 243–247.
[8] D.S. Bernstein, Matrix Mathematics: Theory, Facts, and Formulas, Princeton University Press, New Jersey, 2009.
[9] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[10] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
[11] A. Chandrashekaran, T. Parthasaraty, V. Vetrivel, On the $P_2'$ and $P_2$-properties in the semidefinite linear complementarity problem, Linear Algebra Appl. 432 (2010) 134–143.
[12] M.S. Gowda, Y. Song, G. Ravindran, On some interconnections between strict monotonicity, globally uniquely solvable, and $P$ properties in semidefinite linear complementarity problems, Linear Algebra Appl. 370 (2003) 355–368.
[13] M.S. Gowda, R. Sznajder, Some global and solvability results for linear complementarity problems over symmetric cones, SIAM J. Optim. 18 (2007) 461–481.
[14] M.S. Gowda, J. Tao, Z-transformations on proper and symmetric cones, Math. Program. 117 (2009) 195–221.
[15] A. Auslender, M. Teboulle, Asymptotic Cones and Functions in Optimization and Variational Inequalities, Springer, Berlin, 2003.
[16] F. Facchinei, J.-S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. I, Springer-Verlag, New York, 2003.
[17] M.S. Gowda, T.I. Seidman, Generalized linear complementarity problems, Math. Program. 46 (1990) 329–340.
[18] M.S. Gowda, R. Sznajder, J. Tao, Some $P$-properties for linear transformations on Euclidean Jordan algebras, Linear Algebra Appl. 393 (2004) 203–232.
[19] J. Faraut, A. Korányi, Analysis on Symmetric Cones, Clarendon Press, Oxford, 1994.
[20] R. Sznajder, M.S. Gowda, M.M. Moldovan, More results on Schur complements in Euclidean Jordan algebras, J. Global Optim. (2011) 1–14.
[21] D. Sampangi Raman, Some contributions to semidefinite linear complementarity problems, Ph.D. Thesis, Indian Statistical Institute, Chennai, 2004.
[22] M.S. Gowda, Pseudomonotone and copositive star matrices, Linear Algebra Appl. 113 (1989) 107–118.
[23] M.S. Gowda, T. Parthasarathy, Complementarity forms of theorems of Lyapunov and Stein, and related results, Linear Algebra Appl. 320 (2000) 131–144.