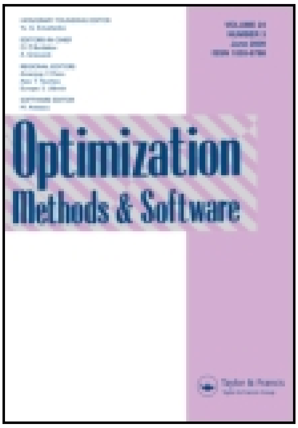

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Interior proximal algorithm with variable metric for second-order cone programming: applications to structural optimization and support vector machines

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Interior proximal algorithm with variable metric for second-order cone programming: applications to structural optimization and support vector machines

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In this work, we propose an inexact interior proximal-type algorithm for solving convex second-order cone programs. This kind of problem consists of minimizing a convex function (possibly nonsmooth) over the intersection of an affine linear space with the Cartesian product of second-order cones. The proposed algorithm uses a variable metric, which is induced by a class of positive-definite matrices and an appropriate choice of regularization parameter. This choice ensures the well definedness of the proximal algorithm and forces the iterates to belong to the interior of the feasible set. Also, under suitable assumptions, it is proven that each limit point of the sequence generated by the algorithm solves the problem. Finally, computational results applied to structural optimization and support vector machines are presented.

Keywords: proximal method; second-order cone programming; variable metric; structural optimization; multiload model; support vector machines; robust classifier

1. Introduction

In this paper, we consider the following convex *second-order cone programming* (SOCP) problem

$$\text{(SOCP)} \quad f_* = \min_{x \in \mathbb{R}^n} f(x); \quad \mathbf{B}x = \mathbf{d}, \quad w^j(x) = A^j x + b^j \in \mathcal{L}_+^{m_j}, \quad j = 1, \dots, J,$$

where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function (possibly nonsmooth), $\mathbf{B}$ is a full rank $r \times n$ real matrix with $r \leq n$, $\mathbf{d} \in \mathbb{R}^r$, A^j are full rank $m_j \times n$ real matrices, and $b^j \in \mathbb{R}^{m_j}$, $j = 1, \dots, J$. For an integer $m \geq 2$, the set $\mathcal{L}_+^m$ denotes the second-order cone (SOC) (also called the Lorentz cone or ice-cream cone) of dimension m defined as $\mathcal{L}_+^m = \{y = (y_1, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{m-1} : \|\bar{y}\| \leq y_1\}$, where $\|\cdot\|$ denotes the Euclidean norm. Since the norm is not differentiable at 0, (SOCP) is not in the class of smooth convex programs. On the other hand, a Lorentz cone can be rewritten as the smooth nonconvex constraint $\mathcal{L}_+^m = \{y \in \mathbb{R}^m : y_2^2 + \dots + y_m^2 \leq y_1^2, y_1 \geq 0\}$. However, this constraint is not qualified at 0 [17, Definition 3.20].

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In recent years, SOCP has received considerable attention because of its wide range of applications in engineering, control and robust optimization (see, for instance [2,25,35] and the references therein). It is known that $\mathcal{L}_+^m$, like $\mathbb{R}_+^m$ and the cone $\mathcal{S}_+^m$ of $m \times m$ real symmetric positive-semidefinite matrices, belong to the class of symmetric cones to which a Jordan algebra may be associated [15]. Using this connection, interior-point methods have been developed for solving linear programs with SOC constraints [25,37].

In this work, we propose an inexact interior *proximal point algorithm* (PPA) with variable metric for solving a convex SOCP whose objective function is not required to be smooth. The standard PPA was introduced by Martinet [26] based on previous work by Moreau [27], and it was then further developed and studied by Rockafellar [33] for the problem of finding zeros of a maximal monotone operator. Later, several authors [11,12,14] generalized PPA for convex programming with nonnegative constraints, replacing the quadratic regularization term by a Bregman distance or ϕ-divergence distance. Recently, Auslender and Teboulle [6] have dealt with general types of constraints, including SOC and semidefinite ones, via a unified proximal distance framework. In all these works, the pseudo-distances are used to force the iterates to stay in the interior of the feasible set.

The idea of PPA with variable metric was originally studied by Qian [31] for monotone operators and by Bonnans et al. [8] for convex programming [23]. Since then, this idea has been exploited in different articles [9,10]. Oliveira *et al.* [28] considered the matrix $H(x) = \operatorname{diag}(x_1^{-r}, \ldots, x_n^{-r})$, $r \geq 2$, in order to define a variable metric on $\mathbb{R}_+^n$: $\langle \cdot, \cdot \rangle_x^H$, for all $x \in \mathbb{R}_{++}^n$. They defined a new class of variable metric interior PPA for the minimization of a continuous proper convex function on $\mathbb{R}_+^n$. This algorithm uses a regularization parameter appropriately chosen so that the iterates are interior points. Moreover, the convergence to a Karush–Khun–Tucker (KKT) point is obtained.

In this paper, we investigate a variable metric proximal-type algorithm for solving convex SOCP problems, where the metric is induced by a general class of positive-definite matrices, such that the iterates are strictly feasible.The outline of this paper is as follows. In Section 2, we recall some basic notions and properties associated with SOC. In Section 3, we present our algorithm with variable metric and prove its convergence properties. In Section 4, we present the notion of quasi-nonincreasing metrics and we prove the convergence of our method under some suitably chosen assumptions. In Section 5, we describe the case of the metric induced by the Hessian of the spectral logarithm, which is not covered by the analysis in Section 4. Finally, in Section 6, we consider two different applications of linear SOCP (LSOCP), we discuss MATLAB implementations of the proposed algorithms, and we present some computational experiments; this is an intermediate step towards more general and possibly nonsmooth convex problems, which are not addressed in this paper from the numerical point of view.

Notation: For a closed proper convex function f, its effective domain is defined by $\operatorname{dom} f = \{x : f(x) < +\infty\}$ and ∂f denotes its subdifferential [31]. The superscript $\top$ denotes the transpose operator and I_d denotes the identity matrix in $\mathbb{R}^{d \times d}$. For a symmetric matrix M, we denote its smallest and largest eigenvalues by $\boldsymbol{\lambda}_{\min}(M)$ and $\boldsymbol{\lambda}_{\max}(M)$, respectively. Given a matrix $A \in \mathbb{R}^{p \times q}$, the smallest and largest singular value of A will be denoted by $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$, respectively. If we have a finite number of matrices $A^1, \ldots, A^J$ such that each $A^j \in \mathbb{R}^{m_j \times n}$, we define $\boldsymbol{\sigma}_{min}(\mathbf{A}) = \min\{\sigma_{\min}(A^j)\colon j = 1, \ldots, J\}$ and $\mathbf{A} := (A^1; \ldots; A^J) \in \mathbb{R}^{q \times n}$ whose rows are those of A^1 to A^J, where $q = \sum_{j=1}^J m_j$. We also denote by $\mathcal{K} := \mathcal{L}_+^{m_1} \times \cdots \times \mathcal{L}_+^{m_J}$. The set $\mathcal{L}_{++}^m = \{y = (y_1, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{m-1}\colon \|\bar{y}\| < y_1\}$ is the interior of the SOC $\mathcal{K} = \mathcal{L}_+^m$ and the set $\partial \mathcal{L}_+^m = \{y \in \mathcal{L}_+^m : y_1 = \|\bar{y}\|\}$ denotes its boundary. We denote by $\mathbf{X}^*$ the optimal solution set of (SOCP). Finally, we define by $\mathbf{w}(x) := (w^1(x), \ldots, w^J(x)) \in \mathbb{R}^q$, where $w^j(x) = A^j x + b^j$ for $j = 1, \ldots, J$.

2. Algebra preliminaries

Let us recall some basic concepts and properties about the Jordan algebra associated with the SOC $\mathcal{L}_+^m$ with $m \geq 2$ (see [15] for more details). The *Jordan product* of any pair $v = (v_1, \bar{v})$, $w = (w_1, \bar{w}) \in \mathbb{R} \times \mathbb{R}^{m-1}$ is defined by $v \circ w = (v^\top w, v_1\bar{w} + w_1\bar{v})$. This can be written as $v \circ w = \mathrm{Arw}(v)w$, where

$$\mathrm{Arw}(v) := \begin{pmatrix} v_1 & \bar{v}^\top \\ \bar{v} & v_1 I_{m-1} \end{pmatrix}$$

is the *arrow matrix* of v. The bilinear mapping $(v, w) \mapsto v \circ w$ has as the unit element $e = (1, 0, \ldots, 0) \in \mathbb{R}^m$ and is commutative but not associative in general. However, $\circ$ is power associative, that is, for all $w \in \mathbb{R}^m$, w^k can be unambiguously defined as $w^k = w^p \circ w^q$ for any $p, q \in \mathbb{N}$ with $p + q = k$. If $w \in \mathcal{L}_+^m$, then there exists a unique vector in $\mathcal{L}_+^m$, which we denote by $w^{1/2}$, such that $(w^{1/2})^2 = w^{1/2} \circ w^{1/2} = w$.

We next introduce the *spectral factorization* of vectors in $\mathbb{R}^m$ associated with $\mathcal{L}_+^m$. For any $w = (w_1, \bar{w}) \in \mathbb{R} \times \mathbb{R}^{m-1}$, we can decompose w as

$$w = \lambda_1(w)u_1(w) + \lambda_2(w)u_2(w), \tag{1}$$

where $\lambda_i(w)$ and $u_i(w)$ are the *spectral values* and *spectral vectors* of w given by

$$\lambda_i(w) = w_1 + (-1)^i\|\bar{w}\| \quad \text{and} \quad u_i(w) = \begin{cases} \dfrac{1}{2}\left(1, (-1)^i \dfrac{\bar{w}}{\|\bar{w}\|}\right), & \text{if} \quad \bar{w} \neq 0, \\ \dfrac{1}{2}(1, (-1)^i \bar{v}), & \text{if} \quad \bar{w} = 0, \end{cases} \tag{2}$$

for $i = 1, 2$ and $\bar{v}$ being any unit vector in $\mathbb{R}^{m-1}$ (satisfying $\|\bar{v}\| = 1$). Notice that $\lambda_1(w) \leq \lambda_2(w)$ and set $\lambda_{\min}(w) = \lambda_1(w)$, $\lambda_{\max}(w) = \lambda_2(w)$. Some basic properties of these definitions are summarized below [15,16].

Proposition 2.1 *For any $w = (w_1, \bar{w}) \in \mathbb{R} \times \mathbb{R}^{m-1}$, we have*

(a) *If $\bar{w} \neq 0$, then the decomposition* (1) *and* (2) *is unique.*
(b) $\|u_i(w)\| = 1/\sqrt{2}$ *and $u_i(w) \in \partial\mathcal{L}_+^m$ for $i = 1, 2$.*
(c) *$u_1(w)$ and $u_2(w)$ are orthogonal for the Jordan product: $u_1(w) \circ u_2(w) = 0$.*
(d) *$u_i(w)$ is idempotent for the Jordan product: $u_i(w) \circ u_i(w) = u_i(w)$ for $i = 1, 2$.*
(e) *$\lambda_{\min}(w), \lambda_{\max}(w)$ are nonnegative (resp., positive) iff $w \in \mathcal{L}_+^m$ (resp., $w \in \mathcal{L}_{++}^m$).*
(f) *The Euclidean norm of w can be represented as $\|w\|^2 = 1/2(\lambda_{\min}(w)^2 + \lambda_{\max}(w)^2)$.*

The next result provides some interesting inequalities [4, Proposition 3.1].

Proposition 2.2 *Let $v, w \in \mathbb{R}^m$, then $\lambda_{\min}(v) + \lambda_{\min}(w) \leq \lambda_{\min}(v + w) \leq \lambda_{\min}(v) + \lambda_{\max}(w)$, and $\lambda_{\max}(v) + \lambda_{\min}(w) \leq \lambda_{\max}(v + w) \leq \lambda_{\max}(v) + \lambda_{\max}(w)$.*

For each $w = (w_1, \bar{w}) \in \mathbb{R} \times \mathbb{R}^{m-1}$, the trace and determinant of w with respect to $\mathcal{L}_+^m$ are defined as

$$\mathrm{tr}(w) := \lambda_{\min}(w) + \lambda_{\max}(w) = 2w_1; \quad \det(w) := \lambda_{\min}(w)\lambda_{\max}(w) = w_1^2 - \|\bar{w}\|^2. \tag{3}$$

These are the analogues of the trace and determinant of matrices. In order to avoid any misleading, the smallest and largest eigenvalue of a symmetric matrix M are denoted by bold symbols $\boldsymbol{\lambda}_{\min}(M)$ and $\boldsymbol{\lambda}_{\max}(M)$, respectively. A vector $w = (w_1, \bar{w}) \in \mathbb{R} \times \mathbb{R}^{m-1}$ is said to be nonsingular if $\det(w) \neq 0$. If w is nonsingular, then there exists a unique $v = (v_1, \bar{v}) \in \mathbb{R} \times \mathbb{R}^{m-1}$ such that $w \circ v = v \circ w = e$. We call this v the inverse of w and denote it by w^{-1}. Direct calculations yields $w^{-1} = (1/(w_1^2 - \|\bar{w}\|^2))(w_1, -\bar{w}) = (1/\det(w))(\mathrm{tr}(w)e - w)$.

Following [24], for any function $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, we consider the *spectrally defined* function $\Phi_g \colon \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ given by

$$\Phi_g(w) = g(\lambda_{\min}(w)) + g(\lambda_{\max}(w)), \quad \text{if } \lambda_{\min}(w), \lambda_{\max}(w) \in \mathrm{dom}(g) \tag{4}$$

and $\Phi_g(w) = +\infty$ otherwise. If $\lambda_{\min}(w), \lambda_{\max}(w) \in \mathrm{dom}(g)$ then $\Phi_g(w) = \mathrm{tr}(g^{\mathrm{soc}}(w))$ where g^{soc} is the corresponding SOC function defined by

$$g^{\mathrm{soc}}(w) = g(\lambda_{\min}(w))u_1(w) + g(\lambda_{\max}(w))u_2(w), \quad w \in \mathbb{R} \times \mathbb{R}^{m-1}.$$

We have the following result (see [16, Proposition 5.2; 30, Lemma 2.10]).

PROPOSITION 2.3 *Let g be continuously differentiable on int*$(\mathrm{dom}(g)) = \mathbb{R}_{++}$. *Then Φ_g is continuously differentiable on int*$(\mathrm{dom}(\Phi_g)) = \mathcal{L}^m_{++}$ *and for all $w \in \mathcal{L}^m_{++}$, $\nabla\Phi_g(w) = 2(g')^{soc}(w)$. If in addition g' is continuously differentiable in $\mathbb{R}_{++}$ then the Hessian of Φ_g at $w \in \mathcal{L}^m_{++}$ is given by the formula $\nabla^2\Phi_g(w) = 2g''(w_1)I$ if $\bar{w} = 0$, and otherwise is given by*

$$\nabla^2\Phi_g(w) = 2\begin{pmatrix} b & c\bar{w}^\top/\|\bar{w}\| \\ c\bar{w}/\|\bar{w}\| & aI_{m-1} + (b-a)\bar{w}\bar{w}^\top/\|\bar{w}\|^2 \end{pmatrix}, \quad \bar{x}_2 \neq 0,$$

where $a = (g'(\lambda_2) - g'(\lambda_1))/(\lambda_2 - \lambda_1)$, $b = (g''(\lambda_1) + g''(\lambda_2))/2$ and $c = (g''(\lambda_2) - g''(\lambda_1))/2$. If $g''(t) > 0$ for all $t \in \mathbb{R}_{++}$, then $\nabla^2\Phi_g(w)$ is positive-definite for all $w \in \mathcal{L}^m_{++}$.

If, for example, we consider the *logarithm barrier function* $g(t) = -\ln(t)$ with $\mathrm{dom}(g) = \mathbb{R}_{++}$, then its spectrally defined function is given by

$$\Phi_{\ln}(w) = -\ln(w_1^2 - \|\bar{w}\|^2) = -\ln(\det(w)) \quad \text{if } w \in \mathcal{L}^m_{++};\ +\infty \text{ otherwise.}$$

We get $\nabla\Phi_{\ln}(w) = -2w^{-1}$, $w \in \mathcal{L}^m_{++}$. Also, we have an explicit expression for the Hessian of $\Phi_{\ln}$ in $w \in \mathcal{L}^m_{++}$ given by $\nabla^2\Phi_{\ln}(w) = 2(\mathcal{Q}_w)^{-1}$, where

$$\mathcal{Q}_w = \begin{pmatrix} \|w\|^2 & 2w_1\bar{w}^\top \\ 2w_1\bar{w} & \det(w)I_{m-1} + 2\bar{w}\bar{w}^\top \end{pmatrix}. \tag{5}$$

As $g''(t) = 1/t^2 > 0$, it follows that $\mathcal{Q}_w$ is positive-definite $\forall w \in \mathcal{L}^m_{++}$. The matrix $\mathcal{Q}_w$ is called the *quadratic representation* of w, which exists for any $w \in \mathbb{R}^m$. The next result gives some useful properties of $\mathcal{Q}_w$ [2, Theorems 3 and 9].

THEOREM 2.4 *Let $w \in \mathbb{R}^m$ be arbitrary.*

(1) *If w is decomposed as in* (1) *then $\lambda^2_{\min}(w)$ and $\lambda^2_{\max}(w)$ are eigenvalues of $\mathcal{Q}_w$. Furthermore, if $\lambda_{\min}(w) \neq \lambda_{\max}(w)$, then each one has multiplicity* 1. *In addition,* $\det(w)$ *is an eigenvalue of $\mathcal{Q}_w$ and has multiplicity $m-2$ when w is nonsingular and $\lambda_{\min}(w) \neq \lambda_{\max}(w)$.*
(2) *If w is nonsingular, then $\mathcal{Q}_w(\mathcal{L}^m_+) = \mathcal{L}^m_+$; likewise, $\mathcal{Q}_w(\mathcal{L}^m_{++}) = \mathcal{L}^m_{++}$.*

From this theorem, one has in particular that $\mathcal{Q}_w$ is nonsingular if and only if w is nonsingular. The following result is obtained from [37, Proposition 2.1].

LEMMA 2.5 *Let* $w \in \mathcal{L}_+^m$. *Then, there exists a matrix* $\mathcal{Q}_{w^{1/2}}$ *which maps* e *to* w (*that is* $\mathcal{Q}_{w^{1/2}}e = w$), *given explicitly by*

$$\mathcal{Q}_{w^{1/2}} = \begin{pmatrix} w_1 & \bar{w}^\top \\ \bar{w} & \det(w)^{1/2} I_{m-1} + \dfrac{\bar{w}\bar{w}^\top}{\det(w)^{1/2} + w_1} \end{pmatrix}. \tag{6}$$

This matrix is positive-semidefinite and satisfies that $\mathcal{Q}_{w^{1/2}} = \mathcal{Q}_w^{1/2}$. *Moreover, when* $w \in \mathcal{L}_{++}^m$, $\mathcal{Q}_{w^{1/2}}$ *turns out to be a positive-definite matrix. If in addition* $\bar{w} \neq 0$, *then the matrix* $\mathcal{Q}_{w^{1/2}}$ *can be written as follows*:

$$\mathcal{Q}_{w^{1/2}} = Arw(w) - \begin{pmatrix} 0 & 0 \\ 0 & (w_1 - \det(w)^{1/2}) \left(I - \dfrac{\bar{w}\bar{w}^\top}{\|\bar{w}\|^2} \right) \end{pmatrix}. \tag{7}$$

3. Proximal algorithm with variable metric

Let $\mathcal{F} = \{x \in \mathbb{R}^n : w^j(x) = A^j x + b^j \in \mathcal{L}_{++}^{m_j},\ j = 1, \ldots, J\}$, $\mathcal{B} = \{x \in \mathbb{R}^n : \mathbf{B}x = \mathbf{d}\}$ and $C = \mathcal{B} \cap \mathcal{F}$. The feasible set of (SOCP) is $\bar{C}$, the closure of C in $\mathbb{R}^n$. From now on, we suppose that the following assumptions hold true:

(A1) $f_* > -\infty$.
(A2) $\operatorname{dom} f \cap C \neq \emptyset$ (Slater's condition).

3.1 *Algorithm PAVM*

We denote by $\mathbf{M} = \operatorname{diag}(M^1, \ldots, M^J)$ a block diagonal matrix with $M^j \in \mathbb{R}^{m_j \times m_j}$ being symmetric and positive-definite for each $j = 1, \ldots, J$. We suppose that $\mathbf{A}$ has rank n. Set $\langle \cdot, \cdot \rangle_{\mathbf{M}} := \langle \mathbf{A}^\top \mathbf{M}\mathbf{A} \cdot, \cdot \rangle$, and let us define the following induced norms $\|u\|_{\mathbf{M}}^2 := \langle u, u \rangle_{\mathbf{M}} = \langle \mathbf{M}\mathbf{A}u, \mathbf{A}u \rangle$ and $\|u\|_{\mathbf{M}}^{*2} := \langle (\mathbf{A}^\top \mathbf{M}\mathbf{A})^{-1} u, u \rangle$, $\forall u \in \mathbb{R}^n$. The proximal algorithm with variable metric (PAVM) for solving the problem (SOCP) is defined as follows:

For each $k = 1, 2, \ldots$, take $\delta_k > 0$ and $\eta_k > 0$ with $\sum_{k=1}^\infty \delta_k < \infty$ and $\sum_{k=1}^\infty \eta_k < \infty$.

Step 0. Start with some initial point $x^0 \in C$, $g^0 \in \partial f(x^0)$ and block diagonal matrix $\mathbf{M}_0$. Set $k = 0$

Step 1. Given $x^k \in C$, $g^k \in \partial f(x^k)$ and an appropriate matrix $\mathbf{M}_k$ and suitable parameter $\gamma_k > 0$, find x^{k+1}, $g^{k+1} \in \mathbb{R}^n$ and $\omega^{k+1} \in \mathbb{R}^r$ such that

$$g^{k+1} \in \partial f(x^{k+1}), \tag{8}$$

$$g^{k+1} + \gamma_k \mathbf{A}^\top \mathbf{M}_k \mathbf{A}(x^{k+1} - x^k) + \mathbf{B}^\top \omega^{k+1} = \epsilon^{k+1}, \tag{9}$$

$$\mathbf{B}x^{k+1} = \mathbf{d}, \tag{10}$$

where the associated error ϵ^{k+1} satisfies the following conditions:

$$\|\epsilon^{k+1}\| \leq \delta_k, \quad \|\epsilon^{k+1}\| \max(\|x^{k+1}\|, \|x^k\|) \leq \eta_k. \tag{11}$$

Step 2. If x^{k+1} satisfies a prescribed stopping rule, then stop.

Step 3. Update $\mathbf{M}_{k+1}$. Replace k by $k + 1$ and go to step 1.

Remark 3.1 Set $F_k(x) := f(x) + (1/2)\gamma_k\|x - x^k\|^2_{\mathbf{M}_k}$. Since f is a closed proper convex function, it directly follows that F_k has bounded sublevel sets. Therefore, the optimal set of $\inf\{F_k(x) : \mathbf{B}x = \mathbf{d}\}$ is nonempty and compact and (8)–(11) hold with $\epsilon^{k+1} = 0$. Thus, the sequence generated by PAVM is well defined. The second condition on $\{\epsilon^k\}$ in (11) is similar to IPA1 in [6]. This is motivated by the inexact minimization of F_k. Notice that one may compute x^{k+1} by using the *bundle method* or by applying some iterations of a standard descent method for the unconstrained minimization of the strongly convex function F_k, depending on the regularity of f.

Remark 3.2 Note that the matrix $\mathbf{M}_k$ defines the shape of the level curves of the variable metric considered in our PAVM algorithm while the regularization parameter γ_k decides indirectly the step length of the next iterate taking into account this choice of $\mathbf{M}_k$. If we rescale our metric by using $\alpha\mathbf{M}_k$, for some $\alpha > 0$, instead of $\mathbf{M}_k$, this is equivalent to keeping $\mathbf{M}_k$ but replacing γ_k with $\alpha\gamma_k$ in (9).

3.2 *Strictly feasible iterates*

The largest eigenvalue of a block diagonal matrix $\mathbf{M} = \operatorname{diag}(M^1, \dots, M^J)$ is given by $\boldsymbol{\lambda}_{\max}(\mathbf{M}) = \max\{\boldsymbol{\lambda}_{\max}(M^j)\colon j = 1, \dots, J\}$. For any element $\mathbf{z} \in \mathbb{R}^m$, we set $\mathbf{Q}_{\mathbf{z}} := \operatorname{diag}(\mathcal{Q}_{z_1}, \dots, \mathcal{Q}_{z_J})$, where $\mathcal{Q}_{z_j} \in \mathbb{R}^{m_j \times m_j}$ is defined by (5). By virtue of Theorem 2.4, when $z \in \mathcal{L}^m_{++}$ we get $\boldsymbol{\lambda}_{\max}(\mathcal{Q}_z) = \lambda^2_{\max}(z)$, obtaining then

$$\boldsymbol{\lambda}_{\max}(\mathbf{Q}_{\mathbf{z}}) = \max_{j=1,\dots,J}\{\boldsymbol{\lambda}_{\max}(\mathcal{Q}_{z_j})\} = \max\{\lambda^2_{\max}(z_j)\colon j = 1, \dots, J\}. \tag{12}$$

Similarly $\boldsymbol{\lambda}_{\max}(\mathbf{Q}_z^{-1}\mathbf{M}^{-1}) = \max\{\boldsymbol{\lambda}_{\max}(\mathcal{Q}_{z_j}^{-1/2}M^{j^{-1}}\mathcal{Q}_{z_j}^{-1/2})\colon\ j = 1, \dots, J\}$. Analogous definitions can be stated for the smallest eigenvalue $\boldsymbol{\lambda}_{\min}(\cdot)$.

Proposition 3.3 *Suppose that for every $k = 0, 1, \dots$, the parameter γ_k satisfies*

$$\gamma_k > \sqrt{2}(\boldsymbol{\sigma}_{min}(\boldsymbol{A}))^{-1}\boldsymbol{\lambda}_{\max}(\mathbf{Q}_{\boldsymbol{w}(x^k)})^{1/2}\boldsymbol{\lambda}_{\max}(\mathbf{Q}^{-1}_{\boldsymbol{w}(x^k)}\mathbf{M}_k^{-1})[\|g^k\| + \delta_k]. \tag{13}$$

Then the sequence $\{x^k\}$ generated by PAVM is contained in C.

Proof (By induction) This is true for $k = 0$. Now, assume that $x^k \in C$. By construction, x^{k+1} satisfies $\mathbf{B}x^{k+1} = \mathbf{d}$. On the other hand, from the monotonicity of ∂f, it follows that $\langle g^{k+1} - g^k, x^{k+1} - x^k\rangle \geq 0$, which together with (9) yields to

$$\langle \gamma_k\mathbf{A}^\top\mathbf{M}_k\mathbf{A}(x^{k+1} - x^k) + \mathbf{B}^\top\omega^{k+1}, x^{k+1} - x^k\rangle \leq \langle g^k, x^k - x^{k+1}\rangle + \langle \epsilon^{k+1}, x^{k+1} - x^k\rangle.$$

From (10) and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}\gamma_k\langle \mathbf{M}_k\mathbf{A}(x^{k+1} - x^k), \mathbf{A}(x^{k+1} - x^k)\rangle &\leq \|g^k\|\|x^k - x^{k+1}\| + \|\epsilon^{k+1}\|\|x^{k+1} - x^k\|\\ &\leq [\|g^k\| + \delta_k]\|x^{k+1} - x^k\|,\end{aligned} \tag{14}$$

where we used (11). As each M^j_k is a positive-definite matrix, we have that

$$\begin{aligned}\langle \mathbf{M}_k\mathbf{A}(x^{k+1} - x^k), \mathbf{A}(x^{k+1} - x^k)\rangle &= \sum_{j=1}^{J}\langle \mathcal{Q}^{1/2}_{w^j(x^k)}M^j_kA^j(x^{k+1} - x^k), \mathcal{Q}^{-1/2}_{w^j(x^k)}A^j(x^{k+1} - x^k)\rangle\\ &\geq \sum_{j=1}^{J}\boldsymbol{\lambda}_{\min}(\mathcal{Q}^{1/2}_{w^j(x^k)}M^j_k\mathcal{Q}^{1/2}_{w^j(x^k)})\|\mathcal{Q}^{-1/2}_{w^j(x^k)}A^j(x^{k+1} - x^k)\|^2.\end{aligned} \tag{15}$$

By Lemma 2.5, $\mathcal{Q}^{1/2}_{w^j(x^k)}$ is positive-definite. Thus, $\|\mathcal{Q}^{-1/2}_{w^j(x^k)}A^j(x^{k+1}-x^k)\| \geq \boldsymbol{\lambda}_{\min}(\mathcal{Q}^{-1/2}_{w^j(x^k)})\|A^j (x^{k+1}-x^k)\| = \boldsymbol{\lambda}_{\min}(\mathcal{Q}^{-1}_{w^j(x^k)})^{1/2}\|A^j(x^{k+1}-x^k)\|$. Using this lower bound once in (15), it follows that $\langle \mathbf{M}_k\mathbf{A}(x^{k+1}-x^k), \mathbf{A}(x^{k+1}-x^k)\rangle \geq \sum_{j=1}^{J} \boldsymbol{\lambda}_{\min}(\mathcal{Q}^{-1}_{w^j(x^k)})^{1/2}\boldsymbol{\lambda}_{\min}(\mathcal{Q}^{1/2}_{w^j(x^k)}M_k^j\mathcal{Q}^{1/2}_{w^j(x^k)})\| \mathcal{Q}^{-1/2}_{w^j(x^k)}A^j(x^{k+1}-x^k)\|\|A^j(x^{k+1}-x^k)\|$. From (14) and the well-known property $\boldsymbol{\lambda}_{\max}(M^{-1}) = \boldsymbol{\lambda}_{\min}(M)^{-1}$ for any symmetric nonsingular matrix M, it follows that

$$\begin{aligned}
&\gamma_k \sum_{j=1}^{J} \|\mathcal{Q}^{-1/2}_{w^j(x^k)}A^j(x^{k+1}-x^k)\|\|A^j(x^{k+1}-x^k)\| \\
&\quad\leq \boldsymbol{\lambda}_{\max}(\mathbf{Q}_{\mathbf{w}(x^k)})^{1/2}\boldsymbol{\lambda}_{\max}(\mathbf{Q}^{-1}_{\mathbf{w}(x^k)}\mathbf{M}_k^{-1})[\|g^k\|+\delta_k]\|x^{k+1}-x^k\|. \qquad (16)
\end{aligned}$$

Since A^j has full rank, we get $\|A^j(x^{k+1}-x^k)\| \geq 1/\|A^{j\dagger}\|_{\text{spec}}\|x^{k+1}-x^k\|$, where $A^{j\dagger}$ denotes the pseudoinverse of Moore–Penrose of A^j and $\|A\|_{\text{spec}} = \sigma_{\max}(A)$ denotes the spectral norm of a given matrix A. By the identity $\sigma_{\max}(A^{j\dagger}) = (\sigma_{\min}(A^j))^{-1}$ [20, p. 421, Exercise 7], we get from (16) that $\gamma_k\sum_{j=1}^{J}\sigma_{\min}(A^j)\|\mathcal{Q}^{-1/2}_{w^j(x^k)}A^j(x^{k+1}-x^k)\| \leq \boldsymbol{\lambda}_{\max}(\mathbf{Q}_{\mathbf{w}(x^k)})^{1/2}\boldsymbol{\lambda}_{\max}(\mathbf{Q}^{-1}_{\mathbf{w}(x^k)}\mathbf{M}_k^{-1})[\|g^k\|+\delta_k]$, which implies that

$$\begin{aligned}
\sum_{j=1}^{J}\|\mathcal{Q}^{-1/2}_{w^j(x^k)}A^j(x^{k+1}-x^k)\| &\leq \frac{1}{\gamma_k}(\boldsymbol{\sigma}_{min}(\mathbf{A}))^{-1}\boldsymbol{\lambda}_{\max}(\mathbf{Q}_{\mathbf{w}(x^k)})^{1/2}\boldsymbol{\lambda}_{\max}(\mathbf{Q}^{-1}_{\mathbf{w}(x^k)}\mathbf{M}_k^{-1})[\|g^k\|+\delta_k] \\
&< \frac{1}{\sqrt{2}}.
\end{aligned}$$

For the last inequality, we have used (13). On the other hand, it holds from Lemma 2.5 that $\mathcal{Q}^{-1/2}_{w^j(x^k)}w^j(x^k) = e_j$, which yields $\|\mathcal{Q}^{-1/2}_{w^j(x^k)}A^j(x^{k+1}-x^k)\| = \|\mathcal{Q}^{-1/2}_{w^j(x^k)}(w^j(x^{k+1})-w^j(x^k))\| = \|\mathcal{Q}^{-1/2}_{w^j(x^k)}w^j(x^{k+1})-e_j\|$, and by virtue of Proposition 2.1(d), it follows that

$$\|\mathcal{Q}^{-1/2}_{w^j(x^k)}A^j(x^{k+1}-x^k)\| \geq \frac{1}{\sqrt{2}}|\lambda_{\min}(\mathcal{Q}^{-1/2}_{w^j(x^k)}w^j(x^{k+1})-e_j)|$$

for all $j = 1,\ldots,J$. Therefore, for each $j = 1,\ldots,J$, we get $|\lambda_{\min}(\mathcal{Q}^{-1/2}_{w^j(x^k)}w^j(x^{k+1})-e_j)| < 1$, which implies that

$$-1 < \lambda_{\min}(\mathcal{Q}^{-1/2}_{w^j(x^k)}w^j(x^{k+1})-e_j) < 1$$

for all $j = 1,\ldots,J$. By using Weyl's theorem (cf. Proposition 2.2) in both inequalities, we get $0 < \lambda_{\min}(\mathcal{Q}^{-1/2}_{w^j(x^k)}w^j(x^{k+1})) < 2$, $\forall j = 1,\ldots,J$. This implies that $\mathcal{Q}^{-1/2}_{w^j(x^k)}w^j(x^{k+1}) \in \mathcal{L}^{m_j}_{++}$, that is, $w^j(x^{k+1}) \in \mathcal{Q}_{w^j(x^k)^{1/2}}(\mathcal{L}^{m_j}_{++})$ for all $j = 1,\ldots,J$. Therefore, by Theorem 2.4 and (10), it follows that $x^{k+1} \in C$. ■

3.3 *Boundedness and some related results*

Let us recall a technical lemma, which will be useful in the sequel [30].

LEMMA 3.4 (i) *Let* $\{v_k\}$ *and* $\{\alpha_k\}$ *be nonnegative real sequences satisfying* $v_{k+1} \leq v_k + \alpha_k$ *for* $\sum\alpha_k < \infty$. *Then the sequence* $\{v_k\}$ *converges.*

(ii) *Let* $\{\lambda_k\}$ *be a sequence of positive numbers,* $\{a_k\}$ *a real sequence and* $b_n = \sigma_n^{-1}\sum_{k=0}^{n}\lambda_k a_k$, *where* $\sigma_n = \sum_{k=0}^{n}\lambda_k$. *If* $\sigma_n \to \infty$, *one has* $\liminf a_n \leq \liminf b_n \leq \limsup b_n \leq \limsup a_n$.

PROPOSITION 3.5 *Let* $\{x^k\} \subset C$ *be a sequence generated by PAVM under* (13). *Then the following hold*:

(i) $\{f(x^k)\}$ *converges and* $\sum_{k=0}^{\infty}(\gamma_k \sum_{j=1}^{J} \|x^{k+1}-x^k\|^2_{M_k^j}) < \infty$.

(ii) *If* $\boldsymbol{X}^*$ *is nonempty and bounded, then the sequence* $\{x^k\}$ *is bounded.*

Proof

(i) From (9) and (10), and since $g^{k+1} \in \partial f(x^{k+1})$, we have $f(x^k) + \langle \epsilon^{k+1}, x^{k+1}-x^k\rangle \geq f(x^{k+1}) + \gamma_k \sum_{j=1}^{J} \|x^{k+1}-x^k\|^2_{M_k^j} \geq f(x^{k+1})$. By (11), and using $\langle \epsilon^{k+1}, x^{k+1}-x^k\rangle \leq \|\epsilon^{k+1}\|(\|x^k\| + \|x^{k+1}\|) \leq 2\|\epsilon^{k+1}\| \max(\|x^{k+1}\|, \|x^k\|)$, we obtain

$$f(x^{k+1}) + \gamma_k \sum_{j=1}^{J} \|x^{k+1}-x^k\|^2_{M_k^j} \leq f(x^k) + 2\eta_k. \tag{17}$$

Thus, $0 \leq f(x^{k+1}) - f_* \leq f(x^k) - f_* + 2\eta_k$. Hence, using Lemma 3.4(i), we deduce that the sequence $\{f(x^k)\}$ converges. From (17), we get $\sum_{k=0}^{N}(\gamma_k \sum_{j=1}^{J} \|x^{k+1}-x^k\|^2_{M_k^j}) \leq f(x^0) - f(x^{N+1}) + 2\sum_{k=0}^{N} \eta_k \leq f(x^0) - f_* + 2\sum_{k=1}^{N+1} \eta_k$. Letting $N \to +\infty$, we obtain the result.

(ii) Summing (17) over $k = 0, \ldots, l$, one has $f(x^{l+1}) - f(x^0) \leq 2\sum_{k=0}^{l} \eta_k$. Since $\sum_{k=1}^{\infty} \eta_k$ exists, it follows that for some $\bar{\eta} \geq 0$ we have $f(x^{l+1}) \leq f(x^0) + 2\bar{\eta} < \infty$, for all $l \geq 0$. As $\mathbf{X}^*$ is bounded, f is level bounded over $\overline{C}$. Thus, one has that $\{x^k\}$ is a bounded sequence.

■

Remark 3.6 As a consequence of above proposition, it follows that $\{g^k\}$ is bounded when the function f is defined everywhere.

The next result is similar to [12, Lemma 3.2].

LEMMA 3.7 *Let* $\{x^k\}$ *be a sequence generated by PAVM. Then for all* $x \in \bar{C} \cap \operatorname{dom} f$, *the following inequality holds*:

$$\frac{2}{\gamma_k}(f(x^{k+1}) - f(x)) \leq \|x-x^k\|^2_{\mathbf{M}_k} - \|x-x^{k+1}\|^2_{\mathbf{M}_k} - \|x^{k+1}-x^k\|^2_{\mathbf{M}_k} + \frac{2}{\gamma_k}\langle \epsilon^{k+1}, x^{k+1}-x\rangle.$$

Proof For any $x \in \bar{C}$, because $g^{k+1} \in \partial f(x^{k+1})$, we have $f(x^{k+1}) + \langle g^{k+1}, x-x^{k+1}\rangle \leq f(x)$. Using (9) and (10) and the inequality above, we get

$$f(x^{k+1}) - f(x) \leq \langle \epsilon^{k+1}, x^{k+1}-x\rangle - \gamma_k\langle \mathbf{A}^\top \mathbf{M}_k \mathbf{A}(x^{k+1}-x^k), x^{k+1}-x\rangle. \tag{18}$$

Since $\mathbf{M}_k$ is symmetric, we have $\|x-x^k\|^2_{\mathbf{M}_k} = \|x-x^{k+1}\|^2_{\mathbf{M}_k} + \|x^{k+1}-x^k\|^2_{\mathbf{M}_k} + 2\langle \mathbf{A}^\top \mathbf{M}_k \mathbf{A} (x^{k+1}-x^k), x-x^{k+1}\rangle$. Then the result follows directly from (18). ■

4. Quasi-nonincreasing metrics

We consider the following hypotheses on the matrices M_k^j:

(H-i) The sequences $\{M_k^{j^{-1}}\}$ are bounded, for each $j = 1, \ldots, J$.

(H-ii) For each $j = 1, \dots, J$, there exists a nonnegative sequence $\{\nu_k^j\}$ such that $(M_k^j - M_{k+1}^j + \nu_k^j I) \in \mathcal{S}_+^{m_j}$ and $\sum_{k=1}^{\infty} \nu_k^j < \infty$.

Remark 4.1 Since each M_k^j is positive definite, (H-i) is equivalent to saying that there exists a $\underline{\eta}_j > 0$ such that $\boldsymbol{\lambda}_{\min}(M_k^j) > \underline{\eta}_j$, for all $k \in \mathbb{N}$ and $j = 1, \dots, J$. Notice that (H-*ii*) implies that sequences $\{M_k^j\}$ are bounded.

LEMMA 4.2 *Let* $\{x^k\}$ *be a sequence generated by the PAVM under*

$$\gamma_k \geq \sqrt{2}(\boldsymbol{\sigma}_{\min}(\boldsymbol{A}))^{-1}\boldsymbol{\lambda}_{\max}(\mathbf{Q}_{\boldsymbol{w}(x^k)})^{1/2}\boldsymbol{\lambda}_{\max}(\mathbf{Q}^{-1}_{\boldsymbol{w}(x^k)}\mathbf{M}_k^{-1})[\|g^k\| + \delta_k] + \beta_k \tag{19}$$

for some $\beta_k \geq \underline{\beta} > 0$. *Assume that (H-i) holds. Then,* $\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 < \infty$ *and, in particular,* $\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$.

Proof As each M_k^j is positive-definite and each A^j is full rank, one has $\|x^{k+1} - x^k\|^2_{M_k^j} \geq \boldsymbol{\lambda}_{\min}(M_k^j)\|A^j(x^{k+1} - x^k)\|^2 \geq \boldsymbol{\lambda}_{\min}(M_k^j)\sigma_{\min}(A^j)^2\|x^{k+1} - x^k\|^2$, whence $\|x^{k+1} - x^k\|^2_{\mathbf{M}_k} = \sum_{j=1}^{J} \|x^{k+1} - x^k\|^2_{M_k^j} \geq \sum_{j=1}^{J} \boldsymbol{\lambda}_{\min}(M_k^j)\sigma_{\min}(A^j)^2\|x^{k+1} - x^k\|^2$. Now, by the boundedness of the sequence $\{M_k^j\}$ for each $j = 1, \dots, J$, there exists $\underline{\eta}_j > 0$ such that $\boldsymbol{\lambda}_{\min}(M_k^j) > \underline{\eta}_j$, for all $j = 1, \dots, J$. Taking $\underline{\eta} = \underline{\beta} \min_{j=1,\dots,J} \underline{\eta}_j \sigma_{\min}(A^j)^2$, we obtain $\sum_{k=0}^{\infty}(\gamma_k \sum_{j=1}^{J} \|x^{k+1} - x^k\|^2_{M_k^j}) \geq J\underline{\eta} \sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2$, and the result follows from Proposition 3.5(iii). ■

From now on, we define: $\sigma_n = \sum_{j=0}^{n-1} \gamma_j^{-1}$, for all $n \in \mathbb{N}$.

LEMMA 4.3 *Let* $\{x^k\}$ *be the sequence generated by algorithm* (*PAVM*). *Assume that (H-ii) holds. For any* $x \in \bar{C} \cap \operatorname{dom} f$, *the following hold*:

$$-2\sigma_n f(x) + \sum_{k=0}^{n-1} \frac{2}{\gamma_k} f(x^{k+1}) \leq \|x - x^0\|^2_{\mathbf{M}_0} - \|x^n - x\|^2_{\mathbf{M}_n} - \sum_{k=0}^{n-1} \left(\|x^{k+1} - x^k\|^2_{\mathbf{M}_k} - \frac{2}{\gamma_k}\langle \epsilon^{k+1}, x^{k+1} - x\rangle - \sum_{j=1}^{J} \nu_k^j \|A^j(x^{k+1} - x)\|^2 \right). \tag{20}$$

Proof Since $(M_k^j - M_{k+1}^j + \nu_k^j I) \in \mathcal{S}_+^{m_j}$, one has $\|x^{k+1} - x\|^2_{M_k^j} + \nu_k^j \|A(x^{k+1} - x)\|^2 \geq \|x^{k+1} - x\|^2_{M_{k+1}^j}$. By using this inequality in the estimate of Lemma 3.7, we have

$$\frac{2}{\gamma_k}(f(x^{k+1}) - f(x)) \leq \|x - x^k\|^2_{\mathbf{M}_k} - \|x^{k+1} - x\|^2_{\mathbf{M}_{k+1}} - \|x^{k+1} - x^k\|^2_{\mathbf{M}_k} + \frac{2}{\gamma_k}\langle \epsilon^{k+1}, x^{k+1} - x\rangle + \sum_{j=1}^{J} \nu_k^j \|A^j(x^{k+1} - x)\|^2. \tag{21}$$

Summing for $k = 0, \dots, n-1$ the result follows immediately. ■

THEOREM 4.4 *Let* $\{x^k\}$ *be the sequence generated by algorithm (PAVM) under* (19) *for some* $\beta_k \geq \underline{\beta} > 0$. *Assume that (H-ii) holds and that* $\mathbf{X}^*$ *is nonempty and bounded. If* $\lim_{n\to\infty} \sigma_n = +\infty$, *then the following hold*:

(i) *The sequence* $\{f(x^k)\}$ *converges to* f_*.
(ii) *The limit points of* $\{x^k\}$ *belong to* $\mathbf{X}^*$.
(iii) *The sequence* $\{\|x^k - u\|^2_{\mathbf{M}_k}\}$ *converges for all* $u \in \mathbf{X}^*$.
(iv) *Furthermore, if* (H-i) *holds then* $\{x^k\}$ *converges to some* $x^* \in \mathbf{X}^*$.

Proof

(i) Let $\theta_{k+1}(x) = \langle \epsilon^{k+1}, x^{k+1} - x\rangle$ and $\vartheta^j_{k+1}(x) = \nu^j_k \|A^j(x^{k+1} - x)\|^2$ for $j = 1, \ldots, J$. Using (11), one has $\theta_{k+1}(x) \leq \varrho_{k+1}$, where $\varrho_{k+1} = \eta_k + \|x\|\delta_k$, which satisfies $\sum_{k=0}^{\infty} \theta_{k+1}(x) < \infty$ and as $\gamma_k \geq \underline{\beta}$, $\sum_{k=0}^{\infty} \theta_{k+1}(x)\gamma_k^{-1} < \infty$. On the other hand, by boundedness of $\{x^{k+1}\}$ (see Proposition 3.5), there exists $\tau > 0$ such that $\|A^j(x^{k+1} - x)\| \leq \sigma_{\max}(A^j)(\tau + \|x\|)$, for all $j = 1, \ldots, J$ and therefore $\sum_{k=0}^{\infty}\sum_{j=1}^{J} \vartheta^j_{k+1}(x) < \infty$. Then, dividing (20) by σ_n and invoking Lemma 3.4(ii), we get from (20) that $\liminf f(x^n) \leq f(x)$ for each $x \in \bar{C}$ so that $\liminf f(x^n) \leq f_*$, which together with the fact that $f(x^n) \geq f_*$ implies that $\liminf f(x^n) = f_*$. Hence, using Proposition 3.5, it follows that the sequence $\{f(x^k)\}$ converges to f_*.

(ii) From Proposition 3.5, we have that $\{x^k\}$ is bounded. Since f is lsc, passing to the limit and reminding ourselves that $\{x^k\} \subset C$, it follows that each limit point is an optimal solution.

(iii) For all $u \in \mathbf{X}^*$, from inequality (21), we obtain

$$\|x^{k+1} - u\|^2_{\mathbf{M}_{k+1}} \leq \|u - x^k\|^2_{\mathbf{M}_k} + \frac{2}{\gamma_k}\langle \epsilon^{k+1}, x^{k+1} - u\rangle + \sum_{j=1}^{J} \nu^j_k \|A^j(x^{k+1} - u)\|^2.$$

By part (i), we get

$$\|x^{k+1} - u\|^2_{\mathbf{M}_{k+1}} \leq \|u - x^k\|^2_{\mathbf{M}_k} + \frac{2}{\underline{\beta}}\varrho_{k+1} + \sum_{j=1}^{J} \sigma_{\max}(A^j)^2(\tau + \|u\|)^2 \nu^j_k.$$

Then, from the nonnegativity of $\|x^k - u\|^2_{\mathbf{M}_k}$, we can apply Lemma 3.4 for establishing the convergence of $\|x^k - u\|^2_{\mathbf{M}_k}$ for all $u \in \mathbf{X}^*$.

(iv) From part (iii), we obtain that the sequences $\{\|x^k - u\|_{M^j_k}\}$ converge to some $c(u) \in \mathbb{R}^+$, $\forall u \in \mathbf{X}^*$ and for each $j = 1, \ldots, J$. Let x^∞ be a limit point of $\{x^k\}$. Take a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $x^{k_i} \to x^\infty \in \mathbf{X}^*$ (by (ii)). From hypothesis (H-ii), $\{M^j_k\}$ is bounded, for each $j = 1, \ldots, J$. Passing onto a subsequence, if necessary, we can suppose that $M^j_{k_i} \to \overline{M}^j$, for each $j = 1, \ldots, J$. Then $\|x^{k_i} - x^\infty\|^2_{M^j_{k_i}} \to 0$. So that $c(x^\infty) = 0$. Moreover, since $\|x^k - x^\infty\|^2_{M^j_k} \geq \lambda_{\min}(M^j_k)\sigma_{\min}(A^j)^2\|x^k - x^\infty\|^2$ and (H-i) holds true, we get that $x^k \to x^\infty$. ■

The following result yields a global rate of convergence estimate, which is similar to the one obtained for proximal-type algorithms in convex minimization problems.

PROPOSITION 4.5 *Let $\{x^k\}$ be the sequence generated by PAVM. Assume hypotheses (H) hold and that $\boldsymbol{X}^* \neq \emptyset$. Then, there exists $\tau > 0$ such that for all $u \in \boldsymbol{X}^*$, we have*

$$f(x^n) - f(u) \leq \frac{\|u - x^0\|^2_{\mathbf{M}_0} - \|u - x^n\|^2_{\mathbf{M}_n}}{2\sigma_n} - \frac{1}{2\sigma_n}\sum_{k=0}^{n-1}\gamma_k(\sigma_k + \sigma_{k+1})\|x^{k+1} - x^k\|^2_{\mathbf{M}_k}$$
$$+ \frac{1}{2\sigma_n}\sum_{k=0}^{n-1}\left(\frac{2\eta_k + \|u\|\delta_k}{\gamma_k} + 4\sigma_k\eta_{k+1} + \sum_{j=1}^{J}\nu_k^j\sigma^2_{\max}(A^j)(\tau + \|u\|)^2\right). \quad (22)$$

Proof Let $u \in \mathbf{X}^*$. Setting x^k for x in (18), multiplying the resulting inequality by σ_k and using the fact that $\sigma_{k+1} = 1/\gamma_k + \sigma_k$ (with $\sigma_0 = 0$), we get

$$\sigma_{k+1}f(x^{k+1}) - \sigma_k f(x^k) - \frac{1}{\gamma_k}f(x^{k+1}) \leq \sigma_k\langle\epsilon^{k+1}, x^{k+1} - x^k\rangle - \sigma_k\gamma_k\|x^{k+1} - x^k\|^2_{\mathbf{M}_k}.$$

Summing the last inequality over $k = 0, \ldots, n-1$, noting $\sigma_0 = 0$ and using (17), one has

$$\sigma_n f(x^n) - \sum_{k=0}^{n-1}\frac{1}{\gamma_k}f(x^{k+1}) \leq 2\sum_{k=0}^{n-1}\sigma_k\eta_k - \sum_{k=1}^{n-1}\sigma_k\gamma_k\|x^{k+1} - x^k\|^2_{\mathbf{M}_k}. \quad (23)$$

Adding twice (23) to (20), we have

$$2\sigma_n(f(x^n) - f(u)) \leq \|u - x^0\|^2_{\mathbf{M}_0} - \|u - x^n\|^2_{\mathbf{M}_n} - \sum_{k=0}^{n-1}\|x^{k+1} - x^k\|^2_{\mathbf{M}_k}$$
$$- 2\sum_{k=0}^{n-1}\sigma_k\gamma_k\|x^{k+1} - x^k\|^2_{\mathbf{M}_k}$$
$$+ \sum_{k=0}^{n-1}\left(\frac{2}{\gamma_k}\langle\epsilon^{k+1}, x^{k+1} - u\rangle + 4\sigma_k\eta_{k+1} + \sum_{j=1}^{J}\nu_k^j\|A^j(x^{k+1} - u)\|^2\right).$$

Because $\langle\epsilon^{k+1}, x^{k+1} - u\rangle \leq \eta_k + \|u\|\delta_k$, $\|A^j(x^{k+1} - u)\| \leq \sigma_{\max}(A^j)(\tau + \|u\|)$, for some $\tau > 0$, the above inequality can be written as

$$2\sigma_n(f(x^n) - f(u)) \leq \|u - x^0\|^2_{\mathbf{M}_0} - \|u - x^n\|^2_{\mathbf{M}_n} - \sum_{k=0}^{n-1}\gamma_k(\sigma_k + \sigma_{k+1})\|x^{k+1} - x^k\|^2_{\mathbf{M}_k}$$
$$+ \sum_{k=0}^{n-1}\left(\frac{2(\eta_k + \|u\|\delta_k)}{\gamma_k} + 4\sigma_k\eta_{k+1} + \sum_{j=1}^{J}\nu_k^j\sigma^2_{\max}(A^j)(\tau + \|u\|)^2\right).$$

Dividing by $2\sigma_n$, we get the desired inequality. ■

Remark 4.6 Ignoring the negative terms in the estimate of proposition above we obtain

$$f(x^n) - f(u) \leq \frac{\|u - x^0\|^2_{\mathbf{M}_0}}{2\sigma_n}$$
$$+ \frac{1}{\sigma_n}\sum_{k=0}^{n-1}\left(\frac{\eta_k + \|u\|\delta_k}{\gamma_k} + 2\sigma_k\eta_{k+1} + \sum_{j=1}^{J}\frac{\nu_k^j\sigma^2_{\max}(A^j)}{2}(\tau + \|u\|)^2\right).$$

The following result is a direct extension of [18, Theorem 3.1] to our algorithm. For simplicity, we suppose that $\eta_k = \|\epsilon^k\| = 0, \forall k \geq 1$.

THEOREM 4.7 *Let* $\{x^k\}$ *be the sequence generated by PAVM algorithm under* (19) *for some* $\beta_k \geq \underline{\beta} > 0$. *Assume that hypotheses* (H) *hold, that* $\mathbf{X}^*$ *is nonempty and bounded, and that* $\eta_k = \|\epsilon^k\| = 0,\ \forall k \geq 1$. *If* $\lim_{n\to\infty} \sigma_n = +\infty$, *then* $\sigma_n(f(x_n) - f^*) \to 0$.

Proof By Theorem 4.4(iv), x^k converges to some $x^* \in \mathbf{X}^*$. We denote by $\zeta_k = f(x^k) - f(x^*)$. From (17), we have

$$\zeta_k - \zeta_{k+1} = f(x^k) - f(x^{k+1}) \geq \gamma_k \|x^{k+1} - x^k\|^2_{\mathbf{M}_k}. \tag{24}$$

Setting x^* for x in (18), we obtain

$$\begin{aligned}\zeta_{k+1} = f(x^{k+1}) - f(x^*) &\leq -\gamma_k \langle \mathbf{A}^\top \mathbf{M}_k \mathbf{A}(x^{k+1} - x^k), x^{k+1} - x^* \rangle \\ &= -\gamma_k \langle \mathbf{A}^\top \mathbf{M}_k \mathbf{A}(x^{k+1} - x^k), x^k - x^* \rangle - \gamma_k \|x^{k+1} - x^k\|^2_{\mathbf{M}_k} \\ &\leq -\gamma_k \langle \mathbf{A}^\top \mathbf{M}_k \mathbf{A}(x^{k+1} - x^k), x^k - x^* \rangle.\end{aligned}$$

But $|\langle \mathbf{A}^\top \mathbf{M}_k \mathbf{A}(x^{k+1} - x^k), x^k - x^* \rangle| \leq \|x^{k+1} - x^k\|_{\mathbf{M}_k} \|x^* - x^k\|_{\mathbf{M}_k}$, so from the inequality above one has $\zeta_{k+1} \leq \gamma_k \|x^{k+1} - x^k\|_{\mathbf{M}_k} \|x^* - x^k\|_{\mathbf{M}_k}$ or equivalently $\|x^{k+1} - x^k\|_{\mathbf{M}_k} \geq \zeta_{k+1}/\gamma_k \|x^* - x^k\|_{\mathbf{M}_k}$. Using this inequality in (24), we have $\zeta_k \geq \zeta_{k+1} + (\zeta^2_{k+1}/\gamma_k)(\|x^* - x^k\|^2_{\mathbf{M}_k})^{-1} = \zeta_{k+1}(1 + \zeta_{k+1}/\gamma_k \|x^* - x^k\|^2_{\mathbf{M}_k})$, whence

$$\zeta_k^{-1} \leq \zeta_{k+1}^{-1} \left(1 + \frac{\zeta_{k+1}}{\gamma_k \|x^* - x^k\|^2_{\mathbf{M}_k}}\right)^{-1}. \tag{25}$$

On the other hand, setting x^* for x in the estimate of Lemma 3.7, we obtain

$$f(x^{k+1}) \leq f(x^{k+1}) + \frac{\gamma_k}{2} \|x^{k+1} - x^k\|^2_{\mathbf{M}_k} \leq f(x^*) + \frac{\gamma_k}{2} \|x^* - x^k\|^2_{\mathbf{M}_k},$$

which yields to

$$0 \leq \frac{\zeta_{k+1}}{\gamma_k \|x^* - x^k\|^2_{\mathbf{M}_k}} \leq \frac{1}{2}.$$

Moreover, the function $(1 + t)^{-1}$ is convex for $t > -1$, hence $(1 + t)^{-1} \leq 1 - (2/3)t$, for $t \in [0, 1/2]$. This last inequality together with (25) implies that

$$\zeta_k^{-1} \leq \zeta_{k+1}^{-1} \left(1 - \frac{2}{3} \frac{\zeta_{k+1}}{\gamma_k \|x^* - x^k\|^2_{\mathbf{M}_k}}\right) = \zeta_{k+1}^{-1} - \frac{2}{3} \frac{1}{\gamma_k \|x^* - x^k\|^2_{\mathbf{M}_k}}.$$

Summing this for $k = 0, \ldots, n - 1$, we get

$$\zeta_n^{-1} \geq \zeta_n^{-1} - \zeta_0^{-1} \geq \frac{2}{3} \sum_{k=0}^{n-1} \frac{1}{\gamma_k \|x^* - x^k\|^2_{\mathbf{M}_k}},$$

obtaining

$$\zeta_n = f(x^n) - f(x^*) \leq \frac{3}{2} \frac{1}{\sum_{k=0}^{n-1} (\gamma_k \|x^* - x^k\|^2_{\mathbf{M}_k})^{-1}}.$$

Multiplying this inequality by σ_n gives

$$\sigma_n(f(x^n) - f(x^*)) \leq \frac{3}{2}\frac{1}{\sigma_n^{-1}\sum_{k=0}^{n-1}(\gamma_k\|x^* - x^k\|^2_{\mathbf{M}_k})^{-1}}.$$

By Theorem 4.4(iv), we have that $\|x^* - x^k\|^{-1}_{\mathbf{M}_k} \to \infty$. Therefore, using the Silverman–Toeplitz theorem [22, p. 76], the series $\sigma_n^{-1}\sum_{k=0}^{n-1}(\gamma_k\|x^* - x^k\|^2_{\mathbf{M}_k})^{-1} \to \infty$ also. In consequence, the result follows. ■

5. PAVM-Log algorithm

5.1 *Metric induced by the Hessian of the spectral log*

In this section, we consider the following choice for the variable metric matrix:

$$\mathbf{M}_k = 2\mathbf{Q}^{-1}_{\mathbf{w}(x^k)}. \tag{26}$$

This a block diagonal matrix, where each block is given by the inverse of the $m_j \times m_j$ matrix $\mathcal{Q}_{w^j(x)}$ defined in (5). This choice is a natural extension to SOC of the algorithm proposed by Oliveira *et al.* [28]. Notice that (19) reduces to

$$\gamma_k \geq \frac{\sqrt{2}}{2}(\boldsymbol{\sigma}_{\min}(A))^{-1}\boldsymbol{\lambda}_{\max}(\mathbf{Q}_{\mathbf{w}(x^k)})^{1/2}(\|g^k\| + \delta_k) + \beta_k. \tag{27}$$

The algorithm PAVM-Log for solving the problem (SOCP) is as follows.

For each $k = 1, 2, \ldots$, let $\beta_k > 0, \delta_k > 0$ and $\eta_k > 0$ with $\beta_k \in (\underline{\beta}, 1)$ where $\underline{\beta} > 0, \sum\delta_k < \infty$ and $\sum\eta_k < \infty$.

Step 0. Start with some initial point $x^0 \in C$. Set $k = 0$

Step 1. Given $x^k \in C$, $g^k \in \partial f(x^k)$ and γ_k satisfying (27), solve

$$g^{k+1} \in \partial f(x^{k+1}), \tag{28}$$

$$g^{k+1} + 2\gamma_k\mathbf{A}^\top\mathbf{Q}^{-1}_{\mathbf{w}(x^k)}\mathbf{A}(x^{k+1} - x^k) + \mathbf{B}^\top\omega^{k+1} = \epsilon^{k+1}, \tag{29}$$

$$\mathbf{B}x^{k+1} = \mathbf{d}, \tag{30}$$

for some $\omega^{k+1} \in \mathbb{R}^r$, where

$$\|\epsilon^{k+1}\| \leq \delta_k, \quad \|\epsilon^{k+1}\|\max(\|x^{k+1}\|, \|x^k\|) \leq \eta_k. \tag{31}$$

Step 2. If x^{k+1} satisfies a prescribed stopping rule, then stop.

Step 3. Replace k by $k+1$ and go to step 1.

Remark 5.1 By virtue of Proposition 3.5, when $\mathbf{X}^*$ is nonempty and bounded, then $\{\gamma_k\}$ can be chosen to be bounded: it suffices to take the equality in (27).

5.2 *On the convergence of PAVM-Log*

First, notice that Lemma 4.3 and Theorem 4.4 do not apply to PAVM-Log because (H-ii) fails for (26). A similar situation occurs for the interior proximal algorithm proposed by Oliveira *et al.* [28], based on the logarithm barrier on the positive orthant. Following the ideas in [21], the authors of [28] deal with the convergence of their algorithm by showing that any cluster point of the iterates satisfies the KKT stationary conditions of the optimization problem. In our case, the corresponding KKT conditions for (SOCP) are given by [2]:

$$\text{(KKT)}\quad g + \mathbf{B}^\top \omega = \mathbf{A}^\top \mathbf{s}, \quad \mathbf{B}x = \mathbf{d}, \quad \mathbf{w}(x) \in \mathcal{K}, \;\; \mathbf{s} \in \mathcal{K}, \;\; \mathbf{w}(x) \circ \mathbf{s} = 0,$$

where $\mathcal{K} = \mathcal{L}_+^{m_1} \times \cdots \times \mathcal{L}_+^{m_J}$, $\omega \in \mathbb{R}^r$, $g \in \partial f(x)$ and $(x, \mathbf{s}) \in \mathbb{R}^n \times \Pi_{j=1}^J \mathbb{R}^{m_j}$ a pair of primal-dual solutions. Unfortunately, the analysis of [21,23] relies on some componentwise comparison arguments which are not valid for spectral values. This technical problem also arises for several algorithms for SOC and SDP optimization problems. Nevertheless, in our case, we have been able to establish some partial results in the general case and a convergence result when the objective function is linear. To do so, we will need the following technical lemma.

LEMMA 5.2 *For any $s \in \mathbb{R}^m$, we have that $s \in \mathcal{L}_{++}^m$ iff $\langle s, y\rangle > 0$, $\forall y \in \mathcal{L}_+^m$, $y \neq 0$.*

Proof For any $s = (s_1, \bar{s}) \in \mathcal{L}_{++}^m$ and $y = (y_1, \bar{y}) \in \mathcal{L}_+^m$ with $y \neq 0$, we know that $\|\bar{s}\| < s_1$ and $\|\bar{y}\| \leq y_1$. Then $\langle s, y\rangle = s_1 y_1 + \bar{s}^\top \bar{y} \geq s_1 y_1 - \|\bar{s}\|\,\|\bar{y}\| \geq s_1 y_1 - \|\bar{s}\|\, y_1 = y_1(s_1 - \|\bar{s}\|) > 0$, where the first inequality follows from the Cauchy–Schwartz inequality. Now, we suppose that $\langle s, y\rangle > 0, \forall y \in \mathcal{L}_+^m$ with $y \neq 0$. Taking $y = e$ we deduce that $s_1 > 0$. If $\bar{s} = 0$, the result follows. On the other hand, if $\bar{s} \neq 0$, we set $y = (1, -\bar{s}/\|\bar{s}\|)$. It is clear that $y \in \mathcal{L}_+^m$ and $y \neq 0$. Hence, $0 < \langle s, y\rangle = s_1 - \|\bar{s}\| = \lambda_{\min}(s)$. This means that $s \in \mathcal{L}_{++}^m$. ■

PROPOSITION 5.3 *Suppose that f is defined in all $\mathbb{R}^n$ and assume that $\boldsymbol{X}^*$ is nonempty and bounded. Let $\{x^k\}$ be sequence generated by PAVM-Log, then*:

(i) *If a cluster point $\tilde{x}$ of the sequence $\{x^k\}$ belongs to C (i.e. $\tilde{x}$ is strictly feasible), then $\tilde{x}$ is optimal for (SOCP).*

(ii) *The dual sequence $\{\mathbf{s}^{k+1}\}$ defined by*

$$\mathbf{s}^{k+1} := 2\gamma_k \mathbf{Q}^{-1}_{\mathbf{w}(x^k)}(\mathbf{w}(x^k) - \mathbf{w}(x^{k+1})) \tag{32}$$

satisfies

$$\lim_{k\to+\infty} \mathbf{Q}^{1/2}_{\mathbf{w}(x^k)} \mathbf{s}^{k+1} = 0. \tag{33}$$

(iii) *Any cluster point $(\tilde{x}, \tilde{\mathbf{s}}, \tilde{g}, \tilde{\omega})$ of $\{(x^k, \mathbf{s}^k, g^k, \omega^k)\}$ satisfies*

$$\begin{aligned} &\tilde{g} + \mathbf{B}^\top \tilde{\omega} = \mathbf{A}^\top \tilde{\mathbf{s}}, \quad \mathbf{B}\tilde{x} = \mathbf{d}, \;\; \mathbf{w}(\tilde{x}) \in \mathcal{K}, \\ &\lambda_{\max}(\tilde{s}^j) \geq 0 \quad \text{and} \quad w^j(\tilde{x})^\top \tilde{s}^j = 0, \;\; j = 1, \ldots, J. \end{aligned} \tag{34}$$

Proof

(i) Since $\mathbf{X}^*$ is nonempty and bounded and f is defined everywhere, the sequences $\{(x^{k+1}, g^{k+1}, \gamma_k)\}$ are bounded. Thus, there exist a subsequence $\{(x^{k_j+1}, g^{k_j+1}, \gamma_{k_j})\}$ and a point $(\tilde{x}, \tilde{g}, \tilde{\gamma})$ such that $(x^{k_j+1}, g^{k_j+1}, \gamma_{k_j}) \to (\tilde{x}, \tilde{g}, \tilde{\gamma})$ as $j \to +\infty$. Moreover, since $\mathbf{B}$ is onto, the subsequence ω^{k_j} of $\{\omega^k\}$ defined in (29) can be written as $\omega^{k_j+1} = (\mathbf{B}\mathbf{B}^\top)^{-1}\mathbf{B}(\epsilon^{k_j+1} - g^{k_j+1} + \mathbf{A}^\top \mathbf{s}^{k_j+1})$. As $\tilde{x} \in C$, we get that $\lim_{j\to+\infty} \omega^{k_j+1} = -(\mathbf{B}\mathbf{B}^\top)^{-1}\mathbf{B}\tilde{g}$, where we have used Lemma 4.2. Therefore, from (28) it follows that $0 \in \partial f(\tilde{x}) + \mathrm{Im}(\mathbf{B}^\top)$. This condition implies that $\tilde{x}$ is an optimal solution of (SOCP).

(ii) From the definition of $\mathbf{s}^{k+1}$, one has $\gamma_k \|x^{k+1} - x^k\|^2_{\mathbf{M}_k} = 2\gamma_k \langle \mathbf{Q}^{-1}_{\mathbf{w}(x^k)}(\mathbf{w}(x^k) - \mathbf{w}(x^{k+1})), \mathbf{w}(x^k) - \mathbf{w}(x^{k+1})\rangle = 1/2\gamma_k \langle \mathbf{s}^{k+1}, \mathbf{Q}_{\mathbf{w}(x^k)} \mathbf{s}^{k+1}\rangle$. By Remark 5.1, the sequence $\{\gamma_k\}$ can be chosen to be bounded and the conclusion follows from Proposition 3.5.

(iii) By construction of sequence $\{x^k\}$, any cluster point $\tilde{x} \in \mathbb{R}^n$ satisfy $\mathbf{B}\tilde{x} = \mathbf{d}$ and $\mathbf{w}(\tilde{x}) \in \mathcal{K}$. From (29) and (32) it follows that $\mathbf{A}^\top \tilde{\mathbf{s}} = \tilde{g} + \mathbf{B}^\top \tilde{\omega}$, with $\tilde{\mathbf{s}}$ a limit point of the dual sequence $\mathbf{s}^{k+1}$. Moreover, from (33) we get $\lim_{k \to +\infty} \mathbf{Q}^{1/2}_{\mathbf{w}(x^k)} \mathbf{s}^{k+1} = 0$, that is, $\mathbf{Q}^{1/2}_{\mathbf{w}(\tilde{x})} \tilde{\mathbf{s}} = 0$. Now, if $\tilde{\mathbf{s}} = (\tilde{s}^1, \dots, \tilde{s}^J)$ with $\tilde{s}^j \in \mathbb{R}^{m_j}$ for $j = 1, \dots, J$, it follows that $\mathcal{Q}^{1/2}_{w^j(\tilde{x})} \tilde{s}^j = 0$, for $j = 1, \dots, J$ (recall that $\mathbf{Q}^{1/2}_{\mathbf{w}(\tilde{x})} = \operatorname{diag}(\mathcal{Q}^{1/2}_{w^1(\tilde{x})}, \dots, \mathcal{Q}^{1/2}_{w^J(\tilde{x})})$). From (7), we obtain that

$$0 = w^j(\tilde{x}) \circ \tilde{s}^j - \begin{pmatrix} 0 \\ (w_1^j(\tilde{x}) - \det(w^j(\tilde{x}))^{1/2}) \left(\bar{\tilde{s}}^j - \dfrac{\bar{w}^j(\tilde{x})^\top \bar{\tilde{s}}^j}{\|\bar{w}^j(\tilde{x})\|^2} \bar{w}^j(\tilde{x}) \right) \end{pmatrix}; \quad j = 1, \dots, J,$$

where we used that $\tilde{s}^j = (\tilde{s}^j_1, \bar{\tilde{s}}^j) \in \mathbb{R} \times \mathbb{R}^{m_j - 1}$. By the definition of product '$\circ$', the first component in the equation above implies that $w^j(\tilde{x})^\top \tilde{s}^j = 0$, for $j = 1, \dots, J$.

It only remains to prove that $\lambda_{\max}(\tilde{s}^j) \geq 0$ for all $j = 1, \dots, J$. If $w^j(\tilde{x}) \in \mathcal{L}^{m_j}_{++}$ for some $j \in \{1, \dots, J\}$, then $\mathcal{Q}^{1/2}_{w^j(\tilde{x})}$ is nonsingular by [2, Corollary 4] and hence the limit (33) implies that $\tilde{s}^j = 0$ and in particular $\lambda_{\min}(\tilde{s}^j) = \lambda_{\max}(\tilde{s}^j) = 0$. Consider now the case when $w^j(\tilde{x}) \in \partial \mathcal{L}^{m_j}_{+} \setminus \{0\}$ for some $j \in \{1, \dots, J\}$. We argue by contradiction, that is, we suppose that $\lambda_{\min}(\tilde{s}^j) \leq \lambda_{\max}(\tilde{s}^j) < 0$. In that case, by virtue of Proposition 2.1(e), we get $-\tilde{s}^j \in \mathcal{L}^{m_j}_{++}$ and, as $\forall w^j(\tilde{x}) \in \mathcal{L}^{m_j}_{+}$ by Lemma 5.2, it follows that $w^j(\tilde{x})^\top \tilde{s}^j < 0$, which is a contradiction. ■

Remark 5.4 Notice that, as $\mathbf{w}(\tilde{x}) \in \mathcal{K}$ and $\mathbf{w}(\tilde{x})^\top \tilde{\mathbf{s}} = 0$, if $\tilde{\mathbf{s}} = (\tilde{s}^1, \dots, \tilde{s}^J) \in \mathcal{K}$, then we get that $\mathbf{w}(\tilde{x}) \circ \tilde{\mathbf{s}} = 0$ by virtue of [2, Lemma 15]. Hence, in order to verify that the cluster point $(\tilde{x}, \tilde{\mathbf{s}}, \tilde{g}, \tilde{\omega})$ satisfies KKT, it only remains to prove that $\lambda_{\min}(\tilde{s}^j) \geq 0$, which amounts to $\tilde{s}^j \in \mathcal{L}^{m_j}_{+}$, for all $j = 1, \dots, J$. We conjecture that this is true for a general (SOCP).

The following result gives a very special case where we have been able to establish that any cluster point $(\tilde{x}, \tilde{\mathbf{s}}, \tilde{g}, \tilde{\omega})$ of $\{(x^k, \mathbf{s}^k, g^k, \omega^k)\}$ satisfies KKT by showing that $\tilde{\mathbf{s}} \in \mathcal{K}$.

PROPOSITION 5.5 *Under the assumptions and notations of Proposition* 5.3, *if in addition f is supposed to be linear, i.e. $f(x) = c^\top x$, and the following inclusion holds for each $j = 1, \dots, J$*

$$A^j(\operatorname{Ker} \mathbf{B}) \supseteq \mathcal{L}^{m_j}_{+}, \tag{35}$$

then $\tilde{\mathbf{s}} \in \mathcal{K}$. In consequence, any limit point of $\{x^k\}$ satisfies the KKT conditions.

Proof Let $\tilde{\mathbf{s}} = (\tilde{s}^1, \dots, \tilde{s}^J)$ be a limit point of $\{\mathbf{s}^k\}$. Recall that for $j = 1, \dots, J$, $C_j = \{x \in \mathbb{R}^n \colon A^j x + b^j \in \mathcal{L}^{m_j}_{+}\}$, $\mathcal{F} = \prod_{j=1}^J C_j$, $\mathcal{B} = \{x \in \mathbb{R}^n \colon \mathbf{B}x = \mathbf{d}\}$ and $C = \mathcal{B} \cap \mathcal{F}$. It is well known that $\mathbf{X}^*$ be nonempty and bounded iff [5,32]

$$f_\infty(d) > 0, \quad \forall\, d \in C_\infty, \ d \neq 0. \tag{36}$$

Now, note that the recession function of f is given by $f_\infty(d) = c^\top d$, for all $d \in \mathbb{R}^n$, and the recession set of feasible set is given by $C_\infty = \{d \in \mathbb{R}^n \colon A^j d \in \mathcal{L}^{m_j}_{+},\ j = 1, \dots, J,\ \mathbf{B}d = 0\}$. Then condition (36) can be rewritten as $c^\top d > 0, \forall d \neq 0;\ A^j d \in \mathcal{L}^{m_j}_{+}$, $j = 1, \dots, J$, $\mathbf{B}d = 0$. On the other hand, from (34), we get that $c^\top d = (\mathbf{A}^\top \tilde{\mathbf{s}} - \mathbf{B}^\top \tilde{\omega})^\top d$, with $\tilde{\omega}$ limit point of $\{\omega^{k+1}\}$. Thus $\tilde{\mathbf{s}}^\top \mathbf{A} d > 0$, $\forall\, d \neq 0$; $A^j d \in \mathcal{L}^{m_j}_{+}$, $j = 1, \dots, J$, $\mathbf{B}d = 0$. Then (35) implies that $\sum_{j=1}^J v_j^\top \tilde{s}^j = \tilde{\mathbf{v}}^\top \mathbf{s} > 0$, for all $\mathbf{v} \in \mathcal{K}$ with $\mathbf{v} \neq \mathbf{0}$. Fix $j \in \{1, \dots, J\}$ such that $v_j \neq 0$, then Lemma 5.2 implies that $\tilde{s}^j \in \mathcal{L}^{m_j}_{++}$. As this holds for any $j = 1, \dots, J$, it follows from Proposition 5.3 that any limit point of $\{x^k\}$ satisfies the KKT conditions of (SOCP), that is, $\tilde{x} \in \mathbf{X}^*$. ■

6. Computational experiments on some specific applications

6.1 *Preliminaries*

We will discuss some computational results on some specific instances of two classes of SOCP: multiload models in truss structural optimization and robust classification by hyperplanes under data uncertainty. Our goal is to show how our algorithm PAVM-Log works in practice and verify empirically that it produces correct results. On purpose we have chosen two well-known applications that can be formulated as a LSOCP. This allows us to compare our results with those obtained by SeDuMi 1.1R2 toolbox for MATLAB, which implements a primal–dual interior point method for solving LSOCPs [36]. Since by construction our algorithm forces x^k to be strictly feasible, we use SeDuMi's result as a benchmark for the optimal value so that a small difference between $f(x^k)$ and that benchmark will ensure the correctness of our solution, up to some relative error tolerance of course. The computer codes were all written in MATLAB 7.3, Release 2006b. The experiments were performed on a Toshiba Tecra laptop with an Intel Pentium M 740 CPU 1.73 GHz processor and 512 MB of RAM, running Microsoft Windows XP.

6.2 *Truss structural optimization*

A *truss* is a mechanical structure composed of thin elastic bars, connecting some pairs of nodal points in $\mathbb{R}^d$ ($d = 2, 3$). Given a *load* (distribution of external forces), the truss deformates until the reaction forces compensate the external load, storing a certain amount of potential energy, named the *compliance*. This measures the stiffness of the truss, that is, its ability to withstand the load; the less the compliance, the more rigid is the truss with respect to the load [1,7].

Let $n = d \cdot N - s$ be the number of degrees of freedom of a *ground structure* consisting of N nodes, where s is the number of fixed directions. Let $m \geq n$ be the number of potential bars. We denote by $x_i = a_i \ell_i \geq 0$ the volume of the ith bar, where a_i is its cross-sectional area and ℓ_i its length. We assume that external loads $f \in \mathbb{R}^n$ apply only at nodal points and bars are subject to axial tension or compression. The mechanical response of the truss is described by the elastic equilibrium system $K(x)u = f$, where $u \in \mathbb{R}^n$ is the nodal displacements vector and $K(x) = \sum_{i=1}^m x_i K_i$. Here, $x \geq 0$ is the volume vector and $K_i \in \mathbb{R}^{n\times n}$ is the *specific stiffness matrix* of the ith bar, and is given by $K_i = (E_i/\ell_i^2)\zeta_i\zeta_i^\top$, where E_i is the Young modulus for the material of the ith bar and $\zeta_i \in \mathbb{R}^n$ is a vector that contains the cosines and sines describing the orientation of ith bar.

Optimal solutions with respect to compliance using a single load model may be unstable, even under small perturbations in the principal load. An alternative is to consider a *multiload model* instead of the single load one, by minimizing a weighted average of the compliances associated with r different loading scenarios $f_1, \ldots, f_r \in \mathbb{R}^n$ [1,3], namely

$$\begin{aligned} \min_{x\in\mathbb{R}^m, u_j\in\mathbb{R}^n} \frac{1}{2}\sum_{j=1}^r \lambda_j f_j^\top u_j; \quad K(x)u_j = f_j,\ \ j = 1, \ldots, r,\\ \sum_{i=1}^m x_i = V,\ x_i \geq 0, \quad i = 1, \ldots, m, \end{aligned} \tag{37}$$

where $\lambda_j > 0$, $j = 1, \ldots, r$, denote suitable weights on the individual compliance values, for a given volume $V > 0$ of material. As is shown in [25] (see also [7, §3.4.3]), (37) can be equivalently written as the following LSOCP:

$$\min_{x\in\mathbb{R}^m,t_{ij}\in\mathbb{R},y_{ij}\in\mathbb{R}} \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{r}\lambda_j t_{ij};\quad \sum_{i=1}^{m} x_i = V,\quad f_j = \sum_{i=1}^{m} y_{ij}\frac{\sqrt{E_i}}{\ell_i}\zeta_i,\quad j=1,\ldots,r,$$
$$(x_i + t_{ij}, 2y_{ij}, x_i - t_{ij}) \in \mathcal{L}^3_+,\quad i=1,\ldots,m,\quad j=1,\ldots,r. \tag{38}$$

Since the objective function in (38) is linear, the proximal step in PAVM-Log corresponds to the unconstrained stationary condition for a quadratic function, which amounts to solving exactly ($\delta_k = \eta_k = 0$) a linear system of the form:

$$2\gamma_k\mathbf{A}^\top\mathbf{Q}^{-1}_{\mathbf{w}(z^k)}\mathbf{A}z^{k+1} + \mathbf{B}^\top\boldsymbol{\omega}^{k+1} = 2\gamma_k\mathbf{A}^\top\mathbf{Q}^{-1}_{\mathbf{w}(z^k)}\mathbf{A}z^k - \frac{1}{2}(\mathbf{0}_m, \mathbf{1}, \mathbf{0}_m);\quad \mathbf{B}z^{k+1} = \bar{f} \tag{39}$$

for some $\boldsymbol{\omega}^{k+1}\in\mathbb{R}^{rn+1}$, where $z = (\mathbf{x},\mathbf{t},\mathbf{y})\in\mathbb{R}^{(2r+1)m}$ stands for the decision variable with $\mathbf{x} = (x_1,\ldots,x_m)$, $\mathbf{t} = (t_{11}, t_{21},\ldots,t_{m1},\ldots,t_{1r},t_{2r},\ldots,t_{mr})$ and $\mathbf{y} = (y_{11}, y_{21}\ldots, y_{m1},\ldots,y_{1r}, y_{2r},\ldots,y_{mr})$, and

$$w^{ij}(z) = (z_i + z_{jm+i}, 2z_{(r+j)m+i}, z_i - z_{jm+i}) = (x_i + t_{ij}, 2y_{ij}, x_i - t_{ij}).$$

Condition (27) reduces to $\gamma_k \geq (\sqrt{m}/2)\max_{i=1,\ldots,m,j=1,\ldots,k}\{\lambda_{\max}(w^{ij}(z^k))\} + \beta_k$. To speed up convergence, we implemented the following relaxed version: we take the regularization parameter as the smaller of the form

$$\gamma_k(\ell) = \frac{1}{2^\ell}\left[\frac{\sqrt{m}}{2}\max_{i=1,\ldots,m,j=1,\ldots,r}\{\lambda_{\max}(w^{ij}(z^k))\} + \beta_k\right],\quad 0\leq\ell\leq\ell_{\max}, \tag{40}$$

in such a way that the updated proximal point be strictly feasible. More precisely, denote by $z(\ell)$ the proximal point corresponding to the regularization parameter $\gamma_k(\ell)$, that is, $z(\ell)$ is the solution of (39)

$$2\mathbf{A}^\top\mathbf{Q}^{-1}_{\mathbf{w}(z^k)}\mathbf{A}\Delta z^k + \mathbf{B}^\top\tilde{\boldsymbol{\omega}}^{k+1} = -\frac{1}{2}(\mathbf{0}_m, \mathbf{1}, \mathbf{0}_m);\quad \mathbf{B}\Delta z^k = 0$$

for some $\tilde{\boldsymbol{\omega}}^{k+1}\in\mathbb{R}^{rn+1}$. Then we set $z^{k+1} = z(\ell_k^*) = z^k + \gamma_k(\ell_k^*)^{-1}\Delta z^k$, where $\ell_k^* = \max\{0,\ldots,\ell_{\max} : z(\ell)\in C\}$.

Finally, as the stopping rule, we take

$$\frac{\|z^{k+1} - z^k\|}{\|z^{k+1}\|} \leq \text{Tol}, \tag{41}$$

where Tol is a prescribed relative tolerance.

In our experiments, we consider three instances of classic examples of multiload truss optimization: the Michel 2 × 1, the 2D Cantilever and the Dome [1,3]. In Table 1, we summarize some information on the sizes of these problems.

Table 1. Truss design test problems.

Type of problem	No. of bars (m)	No. of nodes (N)	No. of degrees of freedom (n)
Michell 2 × 1	12	6	8
Dome	104	33	75
2D Cantilever	200	82	160

Table 2. Total number of iterations, CPU time, and objective value comparisons for tolerances $\mathrm{Tol} = 10^{-2}, 10^{-3}$.

	$\mathrm{Tol} = 10^{-2}$			$\mathrm{Tol} = 10^{-3}$			
Problem	No. of main iterations	CPU time PAVM	$\frac{c_{\mathrm{PAVM}} - c_{\mathrm{SDM}}}{c_{\mathrm{SDM}}}$	No. of main iterations	CPU time PAVM	$\frac{c_{\mathrm{PAVM}} - c_{\mathrm{SDM}}}{c_{\mathrm{SDM}}}$	CPU time SeDuMi
Michell 2 × 1	09	$00'00''.11$	0.005230	22	$00'00''.16$	0.002425	$00'00''.62$
Dome	10	$00'41''.36$	0.051599	64	$04'29''.48$	0.019733	$00'01''.27$
2D Cantilever	18	$03'07''.28$	0.014832	56	$09'57''.51$	0.005664	$00'01''.40$

The loads applied on the structures Michell 2 × 1 and 2D Cantilever are modelled as two scenarios, one with only horizontal loads and the other one with only vertical loads, with values between 1 and 10 and weights $\lambda = (1/2, 1/2)$. In the case of the Dome, we consider one vertical load and two orthogonal loads which are applied just on the top, with values between 10 and 20, and $\lambda = (1/3, 1/3, 1/3)$. In all cases, $V = 1$. The starting point z^0 is given by $y^0_{\cdot j} = \Gamma_\ell(\Gamma_\ell^\top \Gamma_\ell)^{-1} f_j$, $x_i^0 = 1/m$ and $t_{ij}^0 = y_{ij}^{0^2}/x_i^0 + 2.5$, for $i = 1, \ldots, m$ and $j = 1, \ldots, r$. We take $\beta_k = 0.1$ and $\delta_k = \eta_k = 0$. In (40) we take $\ell_{\max} = 10$. In the stopping rule (41), we use $\mathrm{Tol} = 10^{-2}$ and $\mathrm{Tol} = 10^{-3}$.

Table 2 reports the results of our experiments and provide some comparisons with SeDuMi 1.1R2 toolbox for MATLAB. The second and fifth columns show the number of proximal iterations to fulfil (41), the third and sixth columns report the CPU time by using our MATLAB implementation of PAVM-Log, the fourth and seventh columns provides the relative difference between the value of the objective function (compliance) at the output solution obtained by PAVM-Log algorithm, and the optimal compliance given by SeDuMi, denoted by c_{PAVM} and c_{SDM}, respectively. The last column shows the CPU time required by SeDuMi.

For the 2D Cantilever, the largest problem, PAVM-Log provides output solutions with an optimality gap of 0.6% or 1.5% when compared with the benchmark given by SeDuMi. In the case of the Dome, the same difference varies between 2.0% and 5.2%. With the exception of the Michell 2 × 1, SeDuMi is faster than PAVM-Log. In fact, PAVM-Log's CPU time increases considerably for medium-size problems.

6.3 *Support vector machines under uncertainty*

Let us consider the following general binary classification problem: from some training data points in $\mathbb{R}^n$, each of which belongs to one of two classes, the goal is to determine some way of deciding which class new data points will be in. Suppose that the training data consist of two sets of points whose elements are labelled by either 1 or -1 to indicate the class they belong to. If there exists a strictly separating $(n-1)$-dimensional hyperplane between the two data sets, namely $H(\mathbf{w}, b) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{w}^\top \mathbf{x} - b = 0\}$, then the standard support vector machine (SVM) approach is based on constructing a *linear classifier* according to the function $f(x) = \mathrm{sgn}(\mathbf{w}^\top \mathbf{x} - b)$. As there might be many hyperplanes that classify the data, in order to minimize misclassification, one picks the hyperplane which maximizes the separation (margin) between the two classes, so that the distance from the hyperplane to the nearest data point is maximized. In fact, if we have a set $\mathcal{T} = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)\}$ of m training data points in $\mathbb{R}^n \times \{-1, 1\}$, the *maximum-margin* hyperplane problem can be formulated as the following quadratic programming (QP) optimization problem [13]:

$$\min_{\mathbf{w}, b} \frac{1}{2}\|\mathbf{w}\|^2; \;\; y_i(\mathbf{w}^\top \mathbf{x}_i - b) \geq 1, \quad i = 1, \ldots, m. \tag{42}$$

If this problem is feasible then we say that the training data set $\mathcal{T}$ is *linearly separable*. The linear equations $\mathbf{w}^\top \mathbf{x} - b = 1$ and $\mathbf{w}^\top \mathbf{x} - b = -1$ describe the so-called *supporting* hyperplanes.

Following [34,35], suppose that $\mathbf{X}_1$ and $\mathbf{X}_2$ are random vector variables that generate samples of the positive and negative classes, respectively. In order to construct a maximum-margin linear classifier such that the false-negative and false-positive error rates do not exceed $\eta_1 \in (0, 1]$ and $\eta_2 \in (0, 1]$, respectively, let us consider the following quadratic chance-constrained programming problem:

$$\min_{\mathbf{w},b} \frac{1}{2}\|\mathbf{w}\|^2; \ \mathrm{Prob}\{\mathbf{w}^\top \mathbf{X}_1 - b < 0\} \leq \eta_1, \quad \mathrm{Prob}\{\mathbf{w}^\top \mathbf{X}_2 - b > 0\} \leq \eta_2. \tag{43}$$

In other words, we require that the random variable $\mathbf{X}_i$ lies on the correct side of the hyperplane with probability greater than $1 - \eta_i$ for $i = 1, 2$. Assume that for $i = 1, 2$ we *only know* the mean $\mu_i \in \mathbb{R}^n$ and covariance matrix $\Sigma_i \in \mathbb{R}^{n\times n}$ of the random vector $\mathbf{X}_i$. In this case, for each $i = 1, 2$, we want to be able to classify correctly, up to the rate η_i, even for the *worst distribution* in the class of distributions which have common mean and covariance $\mathbf{X}_i \sim (\mu_i, \Sigma_i)$, replacing the probability constraints in (43) with their *robust* counterparts

$$\sup_{\mathbf{X}_1\sim(\mu_1,\Sigma_1)} \mathrm{Prob}\{\mathbf{w}^\top \mathbf{X}_1 - b < 0\} \leq \eta_1, \qquad \sup_{\mathbf{X}_2\sim(\mu_2,\Sigma_2)} \mathrm{Prob}\{\mathbf{w}^\top \mathbf{X}_2 - b > 0\} \leq \eta_2.$$

By virtue of an appropriate application of the multivariate Chebyshev inequality, this worst distribution approach leads to the following QSOCP, which is a deterministic formulation of (43) (see [34] for all details):

$$\min_{\mathbf{w},b} \frac{1}{2}\|\mathbf{w}\|^2; \ \mathbf{w}^\top \boldsymbol{\mu}_1 - b \geq 1 + \kappa_1\|S_1^\top \mathbf{w}\|, \ b - \mathbf{w}^\top \boldsymbol{\mu}_2 \geq 1 + \kappa_2\|S_2^\top \mathbf{w}\|, \tag{44}$$

where $\Sigma_i = S_i S_i^\top$ (for instance, Cholesky factorization) for $i = 1, 2$, and η_i and κ_i are related via the formula $\kappa_i = \sqrt{(1 - \eta_i)/\eta_i}$. Notice that similar to the standard hard-margin SVM formulation (42), problem (44) can be written as an LSOCP:

$$\min_{t,\mathbf{w},b} t; \ t \geq \|\mathbf{w}\|, \quad \mathbf{w}^\top \boldsymbol{\mu}_1 - b \geq 1 + \kappa_1\|S_1^\top \mathbf{w}\|, \quad b - \mathbf{w}^\top \boldsymbol{\mu}_2 \geq 1 + \kappa_2\|S_2^\top \mathbf{w}\|. \tag{45}$$

Note that any feasible hyperplane must separate the means; hence, the natural condition $\mu_1 \neq \mu_2$ is necessary for (44) to be feasible. Since $\kappa_i \to 0$ when $\eta_i \to 1$, problem (44) can be made feasible whenever $\mu_1 \neq \mu_2$ by choosing appropriate values for η_1 and η_2. By choosing $\eta_1 \neq \eta_2$, this formulation can be used for classification with *preferential bias* towards a particular class; for instance, in the case of medical diagnosis, one can allow a low η_1 and a relatively high η_2 [34, Section 4]. Finally, we can mention that these problems can be unfeasible for some values of η_1 or η_2, for instance, when we take $\eta_i \to 0$, we get $\kappa_i \to \infty$.

So far we have assumed that the mean–covariance pairs (μ_i, Σ_i) are known. However, in many practical situations, we only have the training data set $\mathcal{T} = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)\}$. Assuming that $\mathcal{T}$ consists of two samples of independent observations of the random vectors $\mathbf{X}_1$ for $y = 1$ and $\mathbf{X}_2$ for $y = -1$, the idea is to replace (μ_i, Σ_i) with a statistical estimator $(\widehat{\mu}_i, \widehat{\Sigma}_i)$; this can be done by computing the sample mean and covariance for each class from the available observations.

Finding an initial condition of problem (44) may be difficult. Therefore, we consider the following soft-margin SVM formulation:

$$\begin{aligned} \min_{\mathbf{w},b,\xi} \frac{1}{2}\|\mathbf{w}\|^2 + \nu(\xi_1 + \xi_2); \quad & \mathbf{w}^\top \boldsymbol{\mu}_1 - b \geq 1 - \xi_1 + \kappa_1\|S_1^\top \mathbf{w}\|, \\ b - \mathbf{w}^\top \boldsymbol{\mu}_2 \geq 1 - \xi_2 + \kappa_2\|S_2^\top \mathbf{w}\|, \ & \xi_1 \geq 0, \ \xi_2 \geq 0, \end{aligned} \tag{46}$$

where $\nu > 0$ is a sufficiently large penalty parameter. This is based on the Cortes and Vapnik approach [13] for training data that are not linearly separable. If (44) is feasible, then at the optimum for (46), we obtain $\xi_1 = \xi_2 = 0$; otherwise, we detect unfeasibility. Let us denote the decision variable by $z = (\mathbf{w}, b, \xi_1, \xi_2) \in \mathbb{R}^{n+3}$. Set

$$w^i(z) = (\xi_i + (-1)^{(i+1)}(\mu_i^\top \mathbf{w} - b) - 1, \kappa_i S_i^\top \mathbf{w}) \in \mathcal{L}_+^{n+1},$$

for $i = 1, 2$. As we have also the positivity constraints $\xi_i \geq 0$, we adapt the idea of Oliveira *et al.* [28] to this situation, that is, for those constraints, we consider the Hessian of the logarithm barrier function $\psi(\xi_1, \xi_2) = -\log(\xi_1) - \log(\xi_2)$.

Notice that the objective function is quadratic. Then, the proximal step in PAVM-Log corresponds to the unconstrained stationary condition for a quadratic function, which amounts to solving exactly ($\delta_k = \eta_k = 0$) a linear system. Condition (27) reduces to $\gamma_k \geq (\boldsymbol{\sigma}_{\min}(A))^{-1}/\sqrt{2}\max_{i=1,2}\{\lambda_{\max}(w^i(z^k)), \xi_i^k\}(2\nu^2 + \|\mathbf{w}^k\|)^{1/2} + \beta_k$. As in the previous application, we implement the following relaxed version:

$$\gamma_k(\ell) = \frac{1}{2^\ell}\left[\frac{\sqrt{2}}{2\boldsymbol{\sigma}_{\min}(A)}\max_{i=1,2}\{\lambda_{\max}(w^i(z^k)), \xi_i^k\}(2\nu^2 + \|\mathbf{w}^k\|)^{1/2} + \beta_k\right], \tag{47}$$

and denote by $z(\ell)$ the solution of the corresponding linear system with $\gamma_k(\ell)$ instead of γ_k. Then we set $z^{k+1} = z(\ell_k^*)$, where $\ell_k^* = \max\{0, \dots, \ell_{\max} : z(\ell) \in C\}$.

Let us consider the well-known example called Fisher's Iris data set, which is classical in the pattern recognition literature, see for instance http://archive.ics.uci.edu/ml/datasets/Iris. These data contain two measures taken from a sample of 100 ornamental flowers. The data set contains four attributes of an iris, and the goal is to classify the class of iris based on these four attributes. We restrict ourselves to the two features that contain the most information about the class, namely the petal length and the petal width, sepal length and sepal width. There are three species, *setosa, virginica*, and *versicolor*, of which, two are considered in each set. We look to classify the flowers of each set in two species of the existent ones. In all the examples, $n = 2$.

In our implementation of the PAVM-Log algorithm, we use the following notations and values. We use the MATLAB commands *mean* and *cov* on the training data, to compute the estimated means $\hat{\mu}$ and covariance matrices $\hat{\Sigma}$, respectively. The matrices S_i are computed by Cholesky factorization. The starting point $(\mathbf{w}^0, b^0)$ by means of: $\mathbf{w}^0 = 1.1(\tilde{\mathbf{w}}/\alpha)$, $b^0 = 1.1(\tilde{b}/\alpha)$, where $\alpha = \mu_1^\top \tilde{\mathbf{w}}^0 + \tilde{b}$, $\tilde{\mathbf{w}}^0 = (\mu_{11} - \mu_{21}, \mu_{12} - \mu_{22})$, $\tilde{b}^0 = 1/2(\mu_{21}^2 - \mu_{11}^2 + \mu_{22}^2 - \mu_{12}^2)$. The vector $\tilde{\mathbf{w}}$ is taken as the orthogonal vector to the normal of the segment joining μ_2 and μ_1, $\tilde{b}$ as the value in the hyperplane evaluated in the medium point of μ_2 and μ_1. And, $\xi_i^0 = \kappa(i)\|S_i^\top \mathbf{w}^0\| + 0.9$. We take $\nu = 10,000$, $\beta_k = 0.12$, $\delta_k = \eta_k = 0$. In the relaxed version (47) for the regularization parameter, we take $\ell_{\max} = 10$.

Tables 3 and 4 report the results of our experiments and provide some comparisons with SeDuMi 1.1R2 toolbox for MATLAB. In these tables, the first and second columns show the error rates, the third and sixth columns show the number of iterations to fulfil the stopping rule (41), the fourth and seventh columns report the CPU time by using our implementation in MATLAB of this specialized version of PAVM-Log, the fifth and eighth columns provide the relative difference between the value of the objective function at the output solution obtained by PAVM-Log algorithm, and the optimal given by SeDuMi, denoted by $\mathrm{val}_{\mathrm{PAVM}}$ and $\mathrm{val}_{\mathrm{SDM}}$, respectively. Finally, the last column shows the CPU time required by SeDuMi toolbox using its default configuration. The value × in the table represents infeasibility of the problem. If such a case occurs, CPU times correspond to the time required by the PAVM-Log algorithm to reach the prescribed tolerance, obtaining an infeasible solution (i.e. when $\xi_1 \neq 0$ or $\xi_2 \neq 0$).

In these experiments, we can observe that output solutions are optimal up to a gap whose range varies from 0.1% to 3.0% when compared with the benchmark given by SeDuMi, for different

Table 3. Numerical comparisons with SeDuMi applied to data set: Setosa vs Versicolor.

		Tol = 10^{-4}			Tol = 10^{-5}			
η_1	η_2	No. of main iterations	CPU time	$\frac{\text{val}_{\text{PAVM}} - \text{val}_{\text{SDM}}}{\text{val}_{\text{SDM}}}$	No. of main iterations	CPU time	$\frac{\text{val}_{\text{PAVM}} - \text{val}_{\text{SDM}}}{\text{val}_{\text{SDM}}}$	CPU time SeDuMi
0.7	0.1	27	0″.027	0.006623	82	0″.050	0.006441	0″.210
0.5	0.1	11	0″.031	0.013009	31	0″.031	0.002217	0″.180
0.3	0.1	13	0″.010	×	35	0″.032	×	0″.050
0.1	0.1	7	0″.014	×	20	0″.022	×	0″.070
0.1	0.3	12	0″.022	0.019774	23	0″.029	0.014521	0″.130
0.3	0.3	11	0″.022	0.043908	34	0″.032	0.030113	0″.160
0.3	0.5	18	0″.027	0.021844	51	0″.045	0.017778	0″.200

Table 4. Numerical comparisons with SeDuMi applied to data set: Versicolor vs Virginica.

		Tol = 10^{-4}			Tol = 10^{-5}			
η_1	η_2	No. of main iterations	CPU time	$\frac{\text{val}_{\text{PAVM}} - \text{val}_{\text{SDM}}}{\text{val}_{\text{SDM}}}$	No. of main iterations	CPU time	$\frac{\text{val}_{\text{PAVM}} - \text{val}_{\text{SDM}}}{\text{val}_{\text{SDM}}}$	CPU time SeDuMi
0.9	0.3	7	0″.022	0.015485	10	0″.030	0.010218	0″.170
0.7	0.3	8	0″.022	0.005198	11	0″.023	0.005053	0″.180
0.5	0.3	8	0″.017	0.003034	28	0″.033	0.001307	0″.220
0.3	0.3	7	0″.007	0.004301	19	0″.024	0.004157	0″.210
0.3	0.7	14	0″.018	0.012250	42	0″.029	0.009152	0″.140
0.1	0.3	12	0″.015	×	24	0″.031	×	0″.020
0.7	0.5	7	0″.014	0.019001	12	0″.022	0.015555	0″.150

values of η_i. In all cases, PAVM-Log CPU time is much less than SeDuMi's. Due to the small size of the problems to be solved, we can decrease the error tolerance without much computational cost.

Tables 5 and 6 provide the computational results with smaller tolerances for some values of η_i applied to the first data set, obtaining with PAVM-Log an optimality gap whose range varies from 0.3% to 0.7% with reasonable CPU time.

Table 5. Numerical comparisons with SeDuMi applied to data set: Setosa vs Versicolor.

		Tol = 10^{-6}			Tol = 10^{-7}			
η_1	η_2	No. of Main iterations	CPU time	$\frac{\text{val}_{\text{PAVM}} - \text{val}_{\text{SDM}}}{\text{val}_{\text{SDM}}}$	No. of main iterations	CPU time	$\frac{\text{val}_{\text{PAVM}} - \text{val}_{\text{SDM}}}{\text{val}_{\text{SDM}}}$	CPU time SeDuMi
0.7	0.1	259	0″.173	0.005144	944	0″.608	0.004195	0″.210
0.1	0.3	60	0″.044	0.009389	259	0″.179	0.004317	0″.130
0.3	0.5	143	0″.092	0.008224	492	0″.319	0.004422	0″.200

Table 6. Numerical comparisons with SeDuMi applied to data set: Versicolor vs Virginica.

		Tol = 10^{-6}			Tol = 10^{-7}			
η_1	η_2	No. of main iterations	CPU time	$\frac{\text{val}_{\text{PAVM}} - \text{val}_{\text{SDM}}}{\text{val}_{\text{SDM}}}$	No. of main iterations	CPU time	$\frac{\text{val}_{\text{PAVM}} - \text{val}_{\text{SDM}}}{\text{val}_{\text{SDM}}}$	CPU time SeDuMi
0.7	0.3	14	0″.039	0.004017	19	0″.045	0.003424	0″.180
0.3	0.7	136	0″.092	0.006096	412	0″.301	0.002556	0″.140
0.7	0.5	19	0″.029	0.011935	31	0″.051	0.006506	0″.150

6.4 *Concluding remarks on the numerical tests*

The previous numerical results show that the PAVM-Log algorithm can be applied to solve approximately LSCOP problems. As one should have expected, we see that SeDuMi is much faster in terms of CPU time than PAVM-Log for medium-size LSOCP test problems. For very small size problems, both algorithms are comparable with an optimality gap less than 1%. The comparison with SeDuMi in terms of CPU time is not completely fair because of our rather straightforward implementation of PAVM-Log. Even so, it is natural that in the linear case an interior point method, which is based on self-dual embedding and uses a primal–dual predictor–corrector scheme, performs better than our purely primal proximal-point strategy.

It is worth pointing out that the PAVM-Log algorithm is not intended to compete with numerical methods for LSOCP such as SeDuMi, which is an efficient method in particular for large-scale problems. But PAVM-Log might be considered as an alternative for small-size problems and, more importantly, for nonsmooth convex SOCP for which it is not clear how to extend SeDuMi-like approach. Indeed, convex problems can be addressed by conventional local algorithms since all critical points are global minimizers. The regularized proximal subproblem being strongly convex, we expect local algorithms to perform efficiently enough to find good approximate solutions at reasonable execution time. When the objective function is nonsmooth, we can work with the so-called *bundle methods* [19,23].

In this direction, the computational results presented here should be considered just as an intermediate step towards more general and possibly nonsmooth convex problems, which are not addressed in this paper from the numerical point of view. In fact, the PAVM-Log algorithm as presented here is only schematic. There are a lot of theory aspects and implementation issues which should be addressed before performing and evaluating carefully designed computational experiments in the nonsmooth convex case, and this goes beyond the scope of this paper.

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