J Optim Theory Appl (2018) 176:541–558
https://doi.org/10.1007/s10957-018-1246-0

Metric Subregularity in Generalized Equations

Matthieu Maréchal[1]

Received: 25 November 2016 / Accepted: 10 February 2018 / Published online: 21 February 2018

Abstract In this article, we study the metric subregularity of generalized equations using a new tool of nonsmooth analysis. We obtain a sufficient condition for a generalized equation to be metrically subregular, which is not a necessary condition for metric regularity, using a subtle adjustment of the Mordukhovich coderivative. We apply these results to the study of the metric subregularity in a Cournot duopoly game.

Keywords Metric subregularity · Nonsmooth analysis · Generalized equation

Mathematics Subject Classification 90C31 · 49K40

1 Introduction

Metric subregularity is an important concept in analysis. For this reason, a large quantity of publications about metric subregularity and its applications (see, e.g., [1–8]) exists. This article deals with metric subregularity in generalized equations. Metric subregularity in generalized equations has been studied by many authors (see, e.g., [7–9]) and has many applications; for example, it allows for constraint qualifications in mathematical programming with equilibrium constraints or bi-level problems to be obtained (see, e.g., [10–12]) and it also allows for stability results with respect to a parameter in equilibrium problem to be obtained (see, e.g., [13,14]). In several cases,

Communicated by René Henrion.

✉ Matthieu Maréchal
matthieu.marechal@udp.cl

1 Instituto de Ciencias Basicas, Facultad de Ingenieria y Ciencias, Universidad Diego Portales, Ejercito 441, Santiago, Chile

no metric regularity in a generalized equation exists but metric subregularity does, so the Mordukhovich criterion for metric regularity is not well adapted to the study of metric subregularity in generalized equations. In [15–17], the authors introduce directional versions of metric regularity and metric subregularity and have applied it to nonsmooth optimization problems and MPEC problems.

In some recent advances [1,4–6], the authors introduce some outer objects in order to obtain sufficient conditions for metric subregularity which are not necessary conditions for metric regularity. In this article, we make a subtle adjustment of these outer objects in order to obtain a well adapted sufficient condition for metric subregularity in generalized equations.

The article is organized as follows: in Sect. 2, we introduce the tools of nonsmooth analysis that we used. In Sect. 3, we obtain a sufficient condition for a generalized equation to be metrically subregular. In Sect. 4, we apply the results that we obtained in Sect. 3 in a Cournot duopoly game.

2 Coderivative and Metric Subregularity

This section is directly inspired by Mordukhovich's article [18] and book [19]. We introduce some notations: consider a set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. We define the domain of T by $\mathrm{dom}(T) := \{x \in \mathbb{R}^n \ : \ T(x) \neq \emptyset\}$, the graph of T by $Gr(T) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \ : \ y \in T(x)\}$ and the inverse map of T by $T^{-1}(y) := \{x \in \mathbb{R}^n \ : \ y \in T(x)\}$ for all $y \in \mathbb{R}^m$. We can observe that $Gr(T^{-1}) = \{(y, x) \ : \ (x, y) \in Gr(T)\}$.

Consider a set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Let $\bar{x} \in \mathrm{dom}(T)$. The limsup of T at $\bar{x}$ is given by:

$$\operatorname*{Limsup}_{x \to \bar{x}} T(x) := \left\{x^* \in \mathbb{R}^n \ : \ \exists x_n \to \bar{x}\ ,\ \exists x_n^* \to x^* \text{ with } \forall n\ ,\ x_n^* \in T(x_n)\right\}.$$

Let $K \subset \mathbb{R}^n$ and $x \in \overline{K}$, we define the Fréchet normal cone by:

$$\hat{N}(K, x) := \left\{ x^* \in \mathbb{R}^n \ : \ \operatorname*{Limsup}_{x' \xrightarrow{K} \bar{x}} \frac{\langle x^*, x' - x\rangle}{\|x' - x\|} \leq 0 \right\}.$$

The limiting normal cone of K at $\bar{x} \in \overline{K}$ is defined by:

$$N_L(K, \bar{x}) := \operatorname*{Limsup}_{x \xrightarrow{K} \bar{x}} \hat{N}(K, x).$$

Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in Gr(T)$. The coderivative $D^*T(\bar{x}|\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is given by:

$$\forall y^* \in \mathbb{R}^m,\ D^*T(\bar{x}|\bar{y})(y^*) := \{x^* \in \mathbb{R}^n : (x^*, -y^*) \in N_L(Gr(T), (\bar{x}, \bar{y}))\}.$$

We also define the limiting subdifferential of $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at a point $\bar{x}$ where $\varphi(\bar{x}) < +\infty$ by:

$$\partial\varphi(\bar{x}) := \{x^* \in \mathbb{R}^n \ : \ (x^*, -1) \in N_L(\text{epi}(\varphi), (\bar{x}, \varphi(\bar{x})))\},$$

where $\text{epi}(\varphi) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \ : \ y \geq \varphi(x)\}$.

The following sum rule is very important. The proof is given, for example, in [19, Theorem 6.2]. If $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a set-valued map and $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a C^1 function around $\bar{x}$, then we have, for any $\bar{y} \in F(\bar{x})$:

$$D^*(F + f)(\bar{x}|\bar{y} + f(\bar{x}, \bar{\lambda}))(y^*) = D^*F(\bar{x}|\bar{y})(y^*) + Df(\bar{x})^*(y^*). \tag{1}$$

We recall that $Df(\bar{x})^*$ is the adjoint of the differential $Df(\bar{x})$. When we use the sum rule (1) with $F \equiv O_m$, we obtain: if f is a C^1 function around $\bar{x}$, then $D^*f(\bar{x}|f(\bar{x}))(y^*) = Df(\bar{x})^*(y^*)$.

The following defines the metric regularity and subregularity. We recall that for any subset $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, the distance from x to K is given by $\text{dist}(x, K) := \inf\{\|x - y\| \ : \ y \in K\}$.

Definition 2.1 Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $(\bar{x}, \bar{y}) \in Gr(T)$.

1. We say that T is τ-metrically regular around $(\bar{x}, \bar{y})$, with $\tau > 0$, if there exists a constant $r > 0$ such that:

$$\forall x \in B(\bar{x}, r)\ ,\ \forall y \in B(\bar{y}, r)\ ,\ \tau\text{dist}(x, T^{-1}(y)) \leq \text{dist}(y, T(x)).$$

2. We say that T is τ-metrically subregular at $(\bar{x}, \bar{y})$, with $\tau > 0$, if there exists a constant $r > 0$ such that:

$$\forall x \in B(\bar{x}, r)\ ,\ \tau\text{dist}(x, T^{-1}(\bar{y})) \leq \text{dist}(\bar{y}, T(x)).$$

3. We say that T is metrically regular around $(\bar{x}, \bar{y})$ (resp. T is metrically subregular at $(\bar{x}, \bar{y})$) if there exists a constant $\tau > 0$ such that T is τ-metrically regular around $(\bar{x}, \bar{y})$ (resp. T is τ-metrically subregular at $(\bar{x}, \bar{y})$).

The metric subregularity is a weaker version of the metric regularity. Consider a set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $(\bar{x}, \bar{y}) \in Gr(T)$, the Mordukhovich criterion for metric regularity says that if T has a closed graph, then T is metrically regular around $(\bar{x}, \bar{y})$ if and only if the following implication holds true (see, e.g., [18, 19]):

$$0 \in D^*T(\bar{x}|\bar{y})(y^*) \Longrightarrow y^* = 0. \tag{2}$$

If T is a sum $T = F + f$, where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^1 function, then by sum rule (1), T is metrically regular around $(\bar{x}, \bar{y} + f(\bar{x})) \in Gr(T)$ if and only if the following implication holds true:

$$0 \in D^*F(\bar{x}|\bar{y})(y^*) + Df(\bar{x})^*(y^*) \Longrightarrow y^* = 0. \tag{3}$$

The coderivative gives a criterion for metric regularity which is a sufficient condition of metric subregularity, but in many cases a metric subregular set-valued mapping is not metrically regular. A good tool for metric subregularity is the outer coderivative (see, e.g., [1,4]), which is given by:

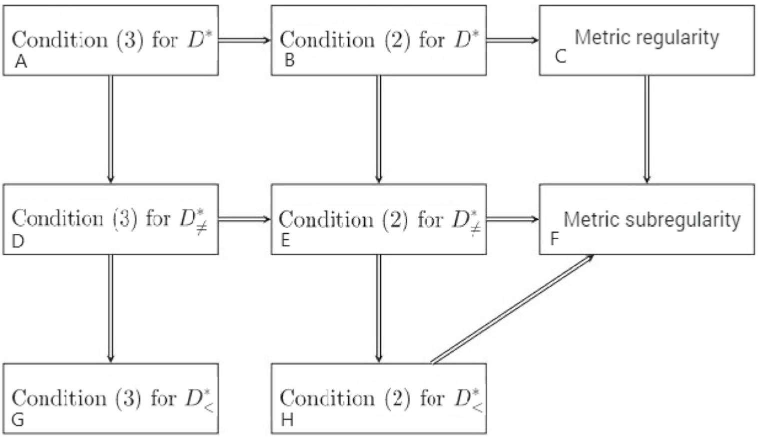
$$D^*_> T(\bar{x}|\bar{y})(y^*) := \left\{ x^* \in \mathbb{R}^n \ : \ (x^*, -y^*) \in \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ (x,y)\in Gr(T) \\ \bar{y}\notin T(x)}} N_L(Gr(T), (x, y)) \right\} \quad (4)$$

In this paper, we introduce the following object, which is a subtle adjustment of the outer coderivative (4).

Definition 2.2 Consider a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$. We define, for any $z^* \in \mathbb{R}^p$, $D^*_{\neq} F(\bar{x}|\bar{z})(z^*)$ by:

$$D^*_{\neq} F(\bar{x}|\bar{z})(z^*) := \left\{ x^* \in \mathbb{R}^n \ : \ (x^*, -z^*) \in \operatorname*{Limsup}_{\substack{(x,z)\to(\bar{x},\bar{z}) \\ (x,z)\in Gr(F) \\ x\neq\bar{x}}} N_L(Gr(T), (x, y)) \right\} \quad (5)$$

Consider a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, a function $f : \mathbb{R}^n \to \mathbb{R}^p$ and $(\bar{x}, \bar{y}) \in \mathrm{Gr}(F)$. Define $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ given by $T(x) = F(x) + f(x)$. We assume that F has a closed graph at $\bar{x}$ and f is C^1 around $\bar{x}$. We claim that we have the following graph:



As we have seen before, the first line results from the sum rule (1) and from [18,19]. Line 1 implies line 2 and line 2 implies line 3 are consequence of the following inclusions:

$$D^*_> T(\bar{x}|\bar{y})(y^*) \subset D^*_{\neq} T(\bar{x}|\bar{y})(y^*) \subset D^* T(\bar{x}|\bar{y})(y^*).$$

Box H implies box F is true by [1, Theorem 3.1]. Box D implies box E is true because the sum rule (1) is also true with $D^*_{\neq}$. Box G cannot imply the metric subregularity as the following example shows.

Example 2.1 Consider $F(x) = \max\{-x, 0\}$ and $f(x) = \max\{0, x\}^2$. We can observe that

$$F(x) + f(x) = \begin{cases} -x, & \text{if } x \leq 0, \\ x^2, & \text{if } x \geq 0. \end{cases}$$

We have $D^*_{>}F(0|0)(y^*) = \{-y^*\}$ and $Df(0)^*(y^*) = 0$, which implies that $D^*_{>}F(0|0)(y^*) + Df(0)^*(y^*) = \{-y^*\}$. The following implication holds,

$$0 \in D^*_{>}F(\bar{x}|\bar{y} - f(\bar{x}))(y^*) + Df(\bar{x})^*(y^*) \Longrightarrow y^* = 0,$$

with $(\bar{x}, \bar{y}) = (0, 0)$, but $F + f$ is not metrically subregular at $(0, 0)$. This proves that the sufficient condition (3) cannot be extended to metric subregularity for the sum with the outer coderivative in place of the coderivative.

The next section deals with the metric subregularity of a set-valued mapping which has the form $(x, y) \rightrightarrows F(x) + f(x, y)$, where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$. The previous graph shows that it may be advantageous to work with the newly proposed $D^*_{\neq}$. We will obtain a sufficient condition for metric subregularity of this class of set-valued mapping in terms of

$$0 \in D^*_{\neq}F(\bar{x}|\bar{z} - f(\bar{x}, \bar{y}))(z^*) \times \{0\} + Df(\bar{x}, \bar{y})^*(z^*) \Longrightarrow z^* = 0. \tag{6}$$

Given the closed subset K of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define:

$$\mathrm{Proj}(x, K) := \{y \in K \;:\; \|x - y\| = \mathrm{dist}(x, K)\}.$$

The following lemma will be used in the proof of Theorem 3.1.

Lemma 2.1 *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping with closed graph and $\bar{y} \in \mathbb{R}^m$. We define $\pi(x) := dist(\bar{y}, T(x))$. Let $\bar{x} \in dom(T) \setminus T^{-1}(\bar{y})$. We have:*

$$\partial\pi(\bar{x}) \subset \bigcup_{y \in Proj(\bar{y}, T(\bar{x}))} D^*T(\bar{x}|y)\left(\|y - \bar{y}\|^{-1}(y - \bar{y})\right).$$

Proof Consider $\sigma(y)$ given by $\sigma(y) = \|y - \bar{y}\|$. We have:

$$\pi(x) = \min\{\sigma(y) \;:\; y \in T(x)\}.$$

Given that σ is a Lipschitzian function, we have, by [19, Theorem 3.38]:

$$\partial\pi(\bar{x}) \subset \bigcup_{y \in Proj(\bar{y}, F(\bar{x}))} \{x^* \in D^*T(\bar{x}|y)(y^*) \;:\; y^* \in \partial\sigma(y)\}.$$

Since $\bar{y} \notin T(\bar{x})$, we have, for any $y \in \text{Proj}(\bar{y}, T(\bar{x}))$, $y \neq \bar{y}$. This implies that $\partial\sigma(y) = \{\|y - \bar{y}\|^{-1}(y - \bar{y})\}$. We then obtain:

$$\partial\pi(\bar{x}) \subset \bigcup_{y \in Proj(\bar{y}, T(\bar{x}))} D^*T(\bar{x}|y)(\|y - \bar{y}\|^{-1}(y - \bar{y})).$$

□

The following lemma relates the coderivative of Φ with the coderivative of F and the differential of f when Φ is given by $\Phi(x, y) = F(x) + f(x, y)$. This lemma is used in the proof of Theorem 3.1.

Lemma 2.2 *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$. We define the set-valued mapping $\Phi : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ by, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$,*

$$\Phi(x, y) := F(x) + f(x, y).$$

Let $(\bar{x}, \bar{y}, \bar{z}) \in Gr(\Phi)$. We suppose that f is a C^1 function around $(\bar{x}, \bar{y})$. We then have:

$$\forall z^* \in \mathbb{R}^p \ , \ D^*\Phi(\bar{x}, \bar{y}|\bar{z})(z^*) = D^*F(\bar{x}|\bar{z} - f(\bar{x}, \bar{y}))(z^*) \times \{0\} + Df(\bar{x}, \bar{y})^*(z^*).$$

Proof By the sum rule on the coderivative of Φ, we obtain that

$$D^*\Phi(\bar{x}, \bar{y}|\bar{z})(z^*) = D^*T(\bar{x}, \bar{y}|\bar{z} - f(\bar{x}, \bar{y}))(z^*) + Df(\bar{x}, \bar{y})^*(z^*),$$

where $T : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ is given by $T(x, y) = F(x)$. Since

$$Gr(T) = \{(x, y, z) \ : \ (x, z) \in Gr(F) \ , \ y \in \mathbb{R}^m\},$$

we obtain that

$$N_L(Gr(T), (\bar{x}, \bar{y}, \bar{z})) = \{(x^*, 0, z^*) \ : \ (x^*, z^*) \in N_L(Gr(F), (\bar{x}, \bar{z}))\},$$

which implies that:

$$D^*T(\bar{x}, \bar{y}|\bar{z} - f(\bar{x}, \bar{y}))(z^*) = D^*F(\bar{x}|\bar{z} - f(\bar{x}, \bar{y}))(z^*) \times \{0\}.$$

We then have:

$$D^*\Phi(\bar{x}, \bar{y}|\bar{z})(z^*) = D^*F(\bar{x}|\bar{z} - f(\bar{x}, \bar{y}))(z^*) \times \{0\} + Df(\bar{x}, \bar{y})^*(z^*).$$

□

3 A Sufficient Condition for Metric Subregularity in a Generalized Equation

In this section, we study the metric subregularity of the set-valued mapping $(x, y) \rightrightarrows F(x) + f(x, y)$ at $(\bar{x}, \bar{y}, \bar{z})$ where $(\bar{x}, \bar{y})$ is a solution of the generalized equation

$$\bar{z} \in F(x) + f(x, y).$$

Theorem 3.1 *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$. We define the set-valued mapping Φ by, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$,*

$$\Phi(x, y) := F(x) + f(x, y).$$

Let $(\bar{x}, \bar{y}, \bar{z}) \in Gr(\Phi)$. Suppose that F has a closed graph near $\bar{x}$, f is a C^1 function around $(\bar{x}, \bar{y})$, the set-valued mapping $y \rightrightarrows F(\bar{x}) + f(\bar{x}, y)$ is τ_1-metrically subregular at $(\bar{y}, \bar{z})$ and

$$\tau_2 := \inf \left\{ \|u^*\| : \begin{array}{l} u^* \in D^*_{\neq} F(\bar{x}|\bar{z} - f(\bar{x}, \bar{y}))(z^*) \times \{0\} + Df(\bar{x}, \bar{y})^*(z^*) \\ \text{with } \|z^*\| = 1 \end{array} \right\} > 0.$$

Then for all $\tau \in]0, \min\{\tau_1, \tau_2\}[$, Φ is τ-metrically subregular at $(\bar{x}, \bar{y}, \bar{z})$.

Proof Let $\tau \in]0, \min\{\tau_1, \tau_2\}[$. We suppose that Φ is not τ-metrically subregular at $(\bar{x}, \bar{y}, \bar{z})$. There exists a sequence $(x_k, y_k) \to (\bar{x}, \bar{y})$ such that for all $k \in \mathbb{N}$,

$$\tau \operatorname{dist}((x_k, y_k), \Phi^{-1}(\bar{z})) > \operatorname{dist}(\bar{z}, \Phi(x_k, y_k)). \tag{7}$$

Clearly this implies that $(x_k, y_k) \notin \Phi^{-1}(\bar{z})$, thus $\bar{z} \notin \Phi(x_k, y_k)$. We introduce the function $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ defined as

$$\forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \ , \ \pi(x, y) = \operatorname{dist}(\bar{z}, \Phi(x, y)).$$

Observe that the inequality (7) can be written as

$$\forall k \in \mathbb{N}, \ \tau \operatorname{dist}((x_k, y_k), \Phi^{-1}(\bar{z})) > \pi(x_k, y_k). \tag{8}$$

We first prove that $\pi(x_k, y_k) \to 0$. Suppose that there exist a subsequence $(x_{k_l}, y_{k_l})_l$ and a constant $L > 0$ such that $\pi(x_{k_l}, y_{k_l}) \geq L$ for all l. Therefore, inequality (8) implies that $\operatorname{dist}((x_{k_l}, y_{k_l}), \Phi^{-1}(\bar{z})) > L/\tau$, which implies that $\|(x_{k_l}, y_{k_l}) - (\bar{x}, \bar{y})\| > L/\tau > 0$. That contradicts the fact that $(x_k, y_k) \to (\bar{x}, \bar{y})$. Therefore, $\pi(x_k, y_k) \to 0$.

Since Φ has a closed graph around $(\bar{x}, \bar{y})$, there exists $\delta > 0$ such that the function π is lower semi-continuous on $\bar{B}((\bar{x}, \bar{y}), \delta)$, then by the variational principle of Ekeland applied with $\varepsilon = \pi(x_k, y_k) > 0$ and $\alpha = \frac{1}{\tau'}\pi(x_k, y_k) > 0$, where $\tau' \in]\tau, \min\{\tau_1, \tau_2\}[$ and k is large enough to ensure that $\frac{1}{\tau'}\pi(x_k, y_k) < \delta$, there exists $(\tilde{x}_k, \tilde{y}_k)$ such that:

$$\|(x_k, y_k) - (\tilde{x}_k, \tilde{y}_k)\| \leq \frac{1}{\tau'}\pi(x_k, y_k), \tag{9}$$

$$\pi(\tilde{x}_k, \tilde{y}_k) \leq \pi(x_k, y_k), \tag{10}$$

$$(\tilde{x}_k, \tilde{y}_k) \in \underset{\bar{B}((\bar{x},\bar{y}),\delta)}{\text{Argmin}}\ \pi(\cdot,\cdot) + \tau'\|(\cdot,\cdot) - (\tilde{x}_k, \tilde{y}_k)\|. \tag{11}$$

We first observe that $(\tilde{x}_k, \tilde{y}_k) \to (\bar{x}, \bar{y})$, indeed by the inequalities (8) and (9), we have

$$\begin{aligned}\|(x_k, y_k) - (\tilde{x}_k, \tilde{y}_k)\| &\leq \frac{1}{\tau'}\pi(x_k, y_k)\\ &\leq \frac{\tau}{\tau'}\text{dist}((x_k, y_k), \Phi^{-1}(\bar{z}))\\ &\leq \frac{\tau}{\tau'}\|(x_k, y_k) - (\bar{x}, \bar{y})\|,\end{aligned}$$

which implies that $\|(x_k, y_k) - (\tilde{x}_k, \tilde{y}_k)\| \to 0$ since $(x_k, y_k) \to (\bar{x}, \bar{y})$, then $(\tilde{x}_k, \tilde{y}_k) \to (\bar{x}, \bar{y})$. We now prove that $(\tilde{x}_k, \tilde{y}_k) \notin \Phi^{-1}(\bar{z})$. By contradiction, if $(\tilde{x}_k, \tilde{y}_k) \in \Phi^{-1}(\bar{z})$, then:

$$\begin{aligned}\pi(x_k, y_k) &< \tau\,\text{dist}((x_k, y_k), \Phi^{-1}(\bar{z})) \quad \text{by (8)}\\ &\leq \tau\|(x_k, y_k) - (\tilde{x}_k, \tilde{y}_k)\|\\ &\leq \frac{\tau}{\tau'}\pi(x_k, y_k) \quad \text{by (9)}.\end{aligned}$$

Therefore, we have $\pi(x_k, y_k) < \frac{\tau}{\tau'}\pi(x_k, y_k)$ which implies that $\tau' < \tau$ (because $(x_k, y_k) \notin \Phi^{-1}(\bar{z})$ then $\pi(x_k, y_k) > 0$ by definition of π), that is a contradiction with $\tau' > \tau$. Finally $(\tilde{x}_k, \tilde{y}_k) \notin \Phi^{-1}(\bar{z})$.

Applying the necessary optimality condition to (11), we deduce that

$$0 \in \partial\pi(\tilde{x}_k, \tilde{y}_k) + \overline{B}\left(0, \tau'\right),$$

then there exists $u_k^* \in \partial\pi(\tilde{x}_k, \tilde{y}_k)$ with $\|u_k^*\| \leq \tau'$. Since $(\tilde{x}_k, \tilde{y}_k) \notin \Phi^{-1}(\bar{z})$, we have $\bar{z} \notin \Phi(\tilde{x}_k, \tilde{y}_k)$, then by Lemma 2.1, we have:

$$u_k^* \in \bigcup_{p \in Proj(\bar{z}, \Phi(\tilde{x_k}, \tilde{y}_k))} D^*\Phi(\tilde{x_k}, \tilde{y}_k|p)\left(\|p - \bar{z}\|^{-1}(p - \bar{z})\right).$$

Let $p_k \in \text{Proj}(\bar{z}, \Phi(\tilde{x_k}, \tilde{y}_k))$ and $z_k^* = \|p_k - \bar{z}\|^{-1}(p_k - \bar{z})$ be such that:

$$u_k^* \in D^*\Phi(\tilde{x}_k, \tilde{y}_k|p_k)(z_k^*). \tag{12}$$

We now prove that $\tilde{x}_k \neq \bar{x}$ for all k large enough. By contradiction, suppose that there exists a subsequence $\tilde{x}_{k_l}$ such that $\tilde{x}_{k_l} = \bar{x}$ for all l large enough. By (11), for all $(x, y) \in B((\bar{x}, \bar{y}), \delta)$, for all $l \in \mathbb{N}$, one has:

$$\pi(\bar{x}, \tilde{y}_{k_l}) \leq \pi(x, y) + \tau'(\|x - x'\| + \|y - \tilde{y}_{k_l}\|). \tag{13}$$

Inequality (13) with $x = \bar{x}$ and $y \in B(\bar{y}, \delta)$ gives:

$$\pi(\bar{x}, \tilde{y}_{k_l}) \leq \pi(\bar{x}, y) + \tau'\|y - \tilde{y}_{k_l}\|. \tag{14}$$

Define the set-valued mapping $\tilde{\Phi} : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ defined as $\tilde{\Phi}(y) := \Phi(\bar{x}, y)$. Let $y \in \tilde{\Phi}^{-1}(\bar{z})$, we have $\pi(\bar{x}, y) = \text{dist}(\bar{z}, \Phi(\bar{x}, y)) = 0$, then by (14), we have $\pi(\bar{x}, \tilde{y}_{k_l}) \leq \tau'\|y - \tilde{y}_{k_l}\|$. Taking the infimum on $y \in \tilde{\Phi}^{-1}(\bar{z})$, we obtain that:

$$\pi(\bar{x}, \tilde{y}_{k_l}) \leq \tau'\text{dist}(\tilde{y}_{k_l}, \tilde{\Phi}^{-1}(\bar{z})). \tag{15}$$

By assumption $\tilde{\Phi}$ is τ_1-metrically subregular at $(\bar{y}, \bar{z})$, then for all l large enough, we have:

$$\tau_1\text{dist}(\tilde{y}_{k_l}, \tilde{\Phi}^{-1}(\bar{z})) \leq \text{dist}(\bar{z}, \tilde{\Phi}(\tilde{y}_{k_l})) = \text{dist}(\bar{z}, \Phi(\bar{x}, \tilde{y}_{k_l})). \tag{16}$$

Using the equality $\pi(\bar{x}, \tilde{y}_{k_l}) = \text{dist}(\bar{z}, \Phi(\bar{x}, \tilde{y}_{k_l}))$, we obtain from (15) and (16) that:

$$\tau_1\text{dist}(\tilde{y}_{k_l}, \tilde{\Phi}^{-1}(\bar{z})) \leq \tau'\text{dist}(\tilde{y}_{k_l}, \tilde{\Phi}^{-1}(\bar{z})).$$

Since $(\bar{x}, \tilde{y}_{k_l}) \notin \Phi^{-1}(\bar{z})$, we have $\tilde{y}_{k_l} \notin \tilde{\Phi}^{-1}(\bar{z})$, then $\text{dist}(\tilde{y}_{k_l}, \tilde{\Phi}^{-1}(\bar{z})) > 0$. Therefore, we obtain $\tau_1 \leq \tau'$ which contradicts the inequality $\tau' < \min\{\tau_1, \tau_2\}$. Finally $\tilde{x}_k \neq \bar{x}$ for all k large enough.

Since for all k, $\|u_k^*\| \leq \tau'$, there exists a subsequence $(u_{k_j}^*)_{j\in\mathbb{N}}$ of $(u_k^*)_{k\in\mathbb{N}}$ such that $u_{k_j}^* \to u^* \in \mathbb{R}^n$. By (12) and by Lemma 2.2, for all j, we have

$$u_{k_j}^* \in D^*F(x_{k_j}|p_{k_j} - f(x_{k_j}, y_{k_j}))(z_{k_j}^*) \times \{0\} + Df(x_{k_j}, y_{k_j})^*(z_{k_j}^*)$$

then

$$u_{k_j}^* - Df(x_{k_j}, y_{k_j})^*(z_{k_j}^*) \in D^*F(x_{k_j}|p_{k_j} - f(x_{k_j}, y_{k_j}))(z_{k_j}^*) \times \{0\}.$$

Given that

$$(x_{k_j}, y_{k_j}) \to (\bar{x}, \bar{y})\ ,\ \ p_{k_j} - f(x_{k_j}, y_{k_j}) \to \bar{z} - f(\bar{x}, \bar{y}) \text{ and } x_{k_j} \neq \bar{x},$$

we have

$$u^* - Df(\bar{x}, \bar{y})^*(z^*) \in D^*_{\neq}F(\bar{x}|\bar{z} - f(\bar{x}, \bar{y}))(z^*) \times \{0\},$$

which implies that

$$u^* \in D^*_{\neq}F(\bar{x}|\bar{z} - f(\bar{x}, \bar{y}))(z^*)\{0\} + Df(\bar{x}, \bar{y})^*(z^*).$$

Since $\|u^*_{k_j}\| \leq \tau'$, for all j, we have $\|u^*\| \leq \tau' < \tau_2$, that is a contradiction with the definition of τ_2. Therefore, Φ is metrically subregular at $(\bar{x}, \bar{y}, \bar{z})$. □

The following Corollary relates the metric subregularity of the set-valued mapping $(x, y) \rightrightarrows F(x) + f(x, y)$ with (6).

Corollary 3.1 *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$. We define the set-valued mapping Φ by, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$,*

$$\Phi(x, y) := F(x) + f(x, y).$$

Let $(\bar{x}, \bar{y}, \bar{z}) \in Gr(\Phi)$. Suppose that F has a closed graph near $\bar{x}$, f is a C^1 function around $(\bar{x}, \bar{y})$, the set-valued mapping $y \rightrightarrows F(\bar{x}) + f(\bar{x}, y)$ is metrically subregular at $(\bar{y}, \bar{z})$ and

$$0 \in D^*_{\neq} F(\bar{x}|\bar{z} - f(\bar{x}, \bar{y}))(z^*) \times \{0\} + Df(\bar{x}, \bar{y})(z^*) \Rightarrow z^* = 0. \tag{17}$$

Then Φ is metrically subregular at $(\bar{x}, \bar{y}, \bar{z})$.

Proof Suppose that Φ is not metrically subregular at $(\bar{x}, \bar{y}, \bar{z})$. Since the set-valued mapping $y \rightrightarrows F(\bar{x}) + f(\bar{x}, y)$ is metrically subregular at $(\bar{y}, \bar{z})$, F has a closed graph near $\bar{x}$ and f is a C^1 function around $(\bar{x}, \bar{y})$, by Theorem 3.1, we have:

$$\inf\left\{\|u^*\| : \begin{array}{l} u^* \in D^*_{\neq} F(\bar{x}|\bar{z} - f(\bar{x}, \bar{y}))(z^*) \times \{0\} + Df(\bar{x}, \bar{y})^*(z^*) \\ \text{with } \|z^*\| = 1 \end{array}\right\} = 0.$$

This implies that there exists a sequence $(x_n, y_n, z_n) \to (\bar{x}, \bar{y}, \bar{z})$ with, for all n, $x_n \neq \bar{x}$ and $(x_n, y_n, z_n) \in Gr(\Phi)$; a sequence z^*_n with norm 1 and a sequence

$$u^*_n \in D^* F(x_n|z_n - f(x_n, y_n))(z^*_n) \times \{0\} + Df(x_n, y_n)^*(z^*_n)$$

with $\|u^*_n\| \to 0$.

Consider a subsequence $z^*_{n_k}$ of z^*_n such that $z^*_{n_k} \to z^*$. We have, for all k, $u^*_{n_k} - Df(x_{n_k}, y_{n_k})^*(z^*_{n_k}) \in D^* F(x_{n_k}|z_{n_k} - f(x_{n_k}, y_{n_k}))(z^*_{n_k}) \times \{0\}$. By definition of $D^*_{\neq}$, since f is C^1 near $(\bar{x}, \bar{y})$ and $x_{n_k} \neq \bar{x}$ for all k, we have

$$-Df(\bar{x}, \bar{y})^*(z^*) \in D^*_{\neq} F(\bar{x}|\bar{z} - f(\bar{x}, \bar{y}))(z^*),$$

which implies that:

$$0 \in D^*_{\neq} F(\bar{x}|\bar{z} - f(\bar{x}, \bar{y}))(z^*) \times \{0\} + Df(\bar{x}, \bar{y})^*(z^*).$$

Then by (17), we have $z^* = 0$, we deduce that $\|z^*_{n_k}\| \to 0$, which contradicts that $\|z^*_{n_k}\| = 1$, for all k. Finally Φ is metrically subregular at $(\bar{x}, \bar{y}, \bar{z})$. □

4 Application to Cournot Duopoly Game

In this section, we apply the previous results to Cournot duopoly game. Cournot duopoly game consists in solving the following generalized Nash equilibrium problem:

$$\begin{aligned}&\text{Player 1: } \min_{x_1} c_1(x_1) - x_1 p(x_1+x_2) \text{ s. t. } 0 \le x_1 \le M_1\\&\text{Player 2: } \min_{x_2} c_2(x_2) - x_2 p(x_1+x_2) \text{ s. t. } 0 \le x_2 \le M_2\end{aligned}$$

where c_1 and c_2 are cost functions and p is the price function. We make the following assumptions:

(H_1) The functions c_1 and c_2 are convex and piecewise C^2 on $[0, +\infty[$.
(H_2) The price function is given by $p(y) = \max(\alpha - \beta y, 0)$ with α and β positive real numbers.
(H_3) For $i \in \{1, 2\}$, there exists a finite set $D(i) \subset]0, M_i[$ such that c_i is C^2 on $\mathbb{R} \setminus D(i)$ and $(c_i)''$ has left and right finite limits at every point of $D(i)$. We define $D(i) = \{a_i^j \;:\; 1 \le j \le q_i\}$.

For each $i \in \{1, 2\}$ and $j \in \{1, \cdots, q_i\}$, we define $b_i^{j,-}$ and $b_i^{j,+}$ such that $\partial c_i(a_i^j) = [b_i^{j,-}, b_i^{j,+}]$ and define:

$$d_i^{j,-} = \lim_{x_i \to a_i^j \;,\; x_i < a_i^j} c_i''(x_i) \;,\; d_i^{j,+} = \lim_{x_i \to a_i^j \;,\; x_i > a_i^j} c_i''(x_i).$$

The first step consists of formulating this problem in terms of a generalized equation. We define $g(x) = \begin{pmatrix} -x_1 \\ x_1 - M_1 \\ -x_2 \\ x_2 - M_2 \end{pmatrix}$. The Cournot duopoly game can be formulated into:

$$\begin{aligned}&\text{Player 1: } \min_{x_1} c_1(x_1) - x_1 p(x_1+x_2) \text{ s. t. } g_1(x) \le 0 \;,\; g_2(x) \le 0.\\&\text{Player 2: } \min_{x_2} c_2(x_2) - x_2 p(x_1+x_2) \text{ s. t. } g_3(x) \le 0 \;,\; g_4(x) \le 0.\end{aligned}$$

We introduce:

$$\begin{aligned}F_1(x) &= \partial c_1(x_1) \times \partial c_2(x_2),\\ F_2(x) &= \begin{pmatrix} -\frac{\partial}{\partial x_1}(x_1 p(x_1+x_2)) \\ -\frac{\partial}{\partial x_2}(x_2 p(x_1+x_2)) \end{pmatrix}\\ &= \begin{pmatrix} -p(x_1+x_2) - x_1 p'(x_1+x_2) \\ -p(x_1+x_2) - x_1 p'(x_1+x_2) \end{pmatrix}\\ &= \begin{pmatrix} -\alpha - \beta(x_1+x_2) - \beta x_1 \\ -\alpha - \beta(x_1+x_2) - \beta x_2 \end{pmatrix},\\ F(x) &= F_1(x) + F_2(x)\end{aligned}$$

and

$$G(x,\lambda)=\begin{pmatrix}\frac{\partial g_1}{\partial x_1}\lambda_1+\frac{\partial g_2}{\partial x_1}\lambda_2\\ \frac{\partial g_3}{\partial x_2}\lambda_3+\frac{\partial g_4}{\partial x_2}\lambda_4\end{pmatrix}=\begin{pmatrix}-\lambda_1+\lambda_2\\ -\lambda_3+\lambda_4\end{pmatrix}.$$

We also define $T(x) = F(x)\times\{0\}$ and $f(x,\lambda)=\begin{pmatrix}G(x,\lambda)\\ \min\{-g(x),\lambda\}\end{pmatrix}$, where $\min\{-g(x),\lambda\} := (\min\{-g_k(x),\lambda_k\})_{k\in\{1,\dots,4\}}$.

Given that the functions $x_1 \mapsto -x_1 p(x_1+x_2)$, $x_2 \mapsto -x_2 p(x_1+x_2)$, c_1 and c_2 are convex, the objective function of each player is convex with respect to its decision variable, then by the KKT conditions, the Cournot duopoly game is equivalent to the following generalized equation:

$$0 \in T(x) + f(x,\lambda). \tag{18}$$

We define $\Phi(x,\lambda) = T(x) + f(x,\lambda)$ and study the metric subregularity of Φ at $(\bar{x},\bar{\lambda},0)$ where $(\bar{x},\bar{\lambda})$ is a solution of (18). In [14, Theorem 6.4], using the coderivative criterion for metric regularity, we obtain that Φ is metrically regular at $(\bar{x}_1,\bar{x}_2,\bar{\lambda},0)$ if $\bar{x}_1\notin D(1)$ or $\bar{x}_2\notin D(2)$. In this section, we use the coderivative, which we define in (5), in order to prove that Φ is metrically subregular at $(\bar{x}_1,\bar{x}_2,\bar{\lambda},0)\in Gr(\Phi)$ without assuming that $\bar{x}_1\notin D(1)$ or $\bar{x}_2\notin D(2)$. We will only consider the case where $\bar{x}_1>0$ and $\bar{x}_2>0$ with $(\bar{x}_1,\bar{x}_2)$ a solution of the Cournot duopoly game because it is natural to assume that both producers produce. We can observe that if c_1 and c_2 are piecewise linear-quadratic then the metric subregularity of Φ is automatic since system (18) becomes polyhedral.

We first prove that given $(\bar{x},\bar{\lambda})$ a solution of (18), the set-valued mapping $\lambda \rightrightarrows \Phi(\bar{x},\lambda)$ is metrically subregular at $(\bar{\lambda},0)$.

Lemma 4.1 *Let $(\bar{x},\bar{\lambda})$ a solution of (18). The set-valued mapping $\lambda\to\Phi(\bar{x},\lambda)$ is metrically subregular at $(\bar{\lambda},0)$.*

Proof We can observe that $\lambda \rightrightarrows \Phi(\bar{x},\lambda)$ is a polyhedral multifunction, then by Robinson's theorem [20], the set-valued mapping $\lambda\to\Phi(\bar{x},\lambda)$ is metrically subregular at $(\bar{\lambda},0)$. □

The following Theorem is a consequence of Corollary 3.1 and the proof is very similar to the proof of Theorem 4.3 in [14]. Given $\bar{x}$, we define

$$A(\bar{x}) := \{k\in\{1,\cdots,4\} \,:\, g_k(\bar{x})=0\}$$

and use the notation A for $A(\bar{x})$ when there is no confusion. Given a vector $y\in\mathbb{R}^q$ and a set $B\subset\{1,\cdots,q\}$, we define y_B as $y_B := (y_k)_{k\in B}$ and $|B|$ as the cardinal of set B.

Theorem 4.1 *Let* $(\bar{x}, \bar{\lambda}, 0) \in Gr(\Phi)$ *with* $(\bar{x}_1, \bar{x}_2) \in]0, M_1] \times]0, M_2]$. *Suppose that* $\bar{\lambda}_A > 0$ *and the following condition holds:*

$$\forall (y^*, z^*) \in \mathbb{R}^2 \times \mathbb{R}^{|A|},$$

$$\left.\begin{array}{l} 0 \in D^*_{\neq} F(\bar{x}| - G(\bar{x}, \bar{\lambda}))(y^*) - Jg_A(\bar{x})^{\perp} z^* \\ y_1^* = 0 \textit{ if } \bar{x}_1 = M_1 \\ y_2^* = 0 \textit{ if } \bar{x}_2 = M_2 \end{array}\right\} \Rightarrow y^* = 0\,,\; z^* = 0. \tag{19}$$

Then the set-valued mapping Φ *is metrically subregular at* $(\bar{z}, 0)$.

Proof Observe that assumption (19) can be written as follows:

$$\left.\begin{array}{l} 0 \in D^*_{\neq} F(\bar{x}| - G(\bar{x}, \bar{\lambda}))(y^*) - Jg_A(\bar{x})^{\perp} z^* \\ Jg_A(\bar{x}) y^* = 0 \end{array}\right\} \Rightarrow y^* = 0\,,\; z^* = 0. \tag{20}$$

We suppose that $0 \in D^*_{\neq} T(\bar{x}| - f(\bar{x}, \bar{\lambda}))(y^*, z^*) \times \{0\} + Jf(\bar{x}, \bar{\lambda})^{\perp} \begin{pmatrix} y^* \\ z^* \end{pmatrix}$. Using the same way than in the proof of Theorem 4.3 in [14] with $D^*_{\neq}$ in place of D^*, we can prove that assumption (20) implies that $y^* = 0$, $z^* = 0$. By Corollary 3.1 and Lemma 4.1, Φ is metrically subregular at $(\bar{z}, 0)$. □

In order to study the metric subregularity of Φ at $(\bar{z}, 0) \in Gr(\Phi)$, we need to compute $D^*_{\neq} F_1(\bar{x}|\bar{y})(y^*)$. Given $\bar{x} \in \mathbb{R}^2$, $\bar{y} \in F_1(\bar{x})$ and $y^* \in \mathbb{R}^2$, for any $i \in \{1, 2\}$, we define $\delta_i(\bar{x}_i, \bar{y}_i, y_i^*)$ as follows: suppose $\bar{x}_i \in D(i)$, let $j \in \{1, \cdots, q_i\}$ such that $\bar{x}_i = a_i^j$, then:

$$\delta_i(\bar{x}_i, \bar{y}_i, y_i^*) = \begin{cases} \emptyset & \text{if } b_i^{j,-} < \bar{y}_i < b_i^{j,+} \text{ and } y_i^* \neq 0, \\ \mathbb{R} & \text{if } y_i^* = 0, \\ [d_i^{j,-} y_i^*, +\infty[& \text{if } \bar{y}_i = b_i^{j,-} \text{ and } y_i^* > 0, \\ \{d_i^{j,-} y_i^*\} & \text{if } \bar{y}_i = b_i^{j,-} \text{ and } y_i^* < 0, \\ ]-\infty, d_i^{j,+} y_i^*] & \text{if } \bar{y}_i = b_i^{j,+} \text{ and } y_i^* < 0, \\ \{d_i^{j,+} y_i^*\} & \text{if } \bar{y}_i = b_i^{j,+} \text{ and } y_i^* > 0. \end{cases}$$

If $\bar{x}_i \notin D(i)$, then we define $\delta_i(\bar{x}_i, \bar{y}_i, y_i^*)$ as $\delta_i(\bar{x}_i, \bar{y}_i, y_i^*) = \{(c_i)''(\bar{x}_i) y_i^*\}$. The following lemma gives $D^* F_1(\bar{x}|\bar{y})(y^*)$, which has been proven in [14].

Lemma 4.2 [14, Lemma 6.1] *Let* $\bar{x} \in \mathbb{R}^2$ *and* $\bar{y} \in F_1(\bar{x})$. *We have:*

$$D^* F_1(\bar{x}|\bar{y})(y^*) = \delta_1(\bar{x}^1, \bar{y}^1, y^{*,1}) \times \delta_2(\bar{x}^2, \bar{y}^2, y^{*,2}).$$

The next lemma gives $D^*_{\neq} F_1(\bar{x}|\bar{y})(y^*)$.

Lemma 4.3 *Let $\bar{x} \in \mathbb{R}^2$ and $\bar{y} \in F_1(\bar{x})$. If $\bar{x}_1 \notin D(1)$ or $\bar{x}_2 \notin D(2)$, then we have*

$$D^*_{\neq} F_1(\bar{x}|\bar{y})(y^*) = \delta_1(\bar{x}_1, \bar{y}_1, y_1^*) \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*),$$

if $\bar{x}_1 \in D(1)$ and $\bar{x}_2 \in D(2)$, there exist $(d_1, d_2) \in [0, +\infty[^2$ such that:

$$D^*_{\neq} F_1(\bar{x}|\bar{y})(y^*) \subset \delta_1(\bar{x}_1, \bar{y}_1, y_1^*) \times \{d_2 y_2^*\} \cup \{d_1 y_1^*\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*).$$

Proof By the definition of $D^*_{\neq} F_1$ given in (5), we have:

$$D^*_{\neq} F_1(\bar{x}|\bar{y})(y^*) = \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ (x,y)\in Gr(F_1) \\ x\neq\bar{x}}} D^* F_1(x|y)(y^*).$$

We consider two cases.

Case 1 $\bar{x}_1 \notin D(1)$ or $\bar{x}_2 \notin D(2)$. Without loss of generality, suppose that $\bar{x}_1 \notin D(1)$. Since c_1 is C^1 at $\bar{x}_1$, we have $c_1'(\bar{x}_1) = \lim_{x_1\to\bar{x}_1} c_1'(x_1)$, then:

$$D^* F_1(\bar{x}|\bar{y})(y^*) = \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ (x,y)\in Gr(F_1) \\ x_1\neq\bar{x}_1}} D^* F_1(x|y)(y^*) \subset D^*_{\neq} F_1(\bar{x}|\bar{y})(y^*).$$

Since by definition we have $D^*_{\neq} F_1(\bar{x}|\bar{y})(y^*) \subset D^* F_1(\bar{x}|\bar{y})(y^*)$, we then have equality, thus we can conclude by Lemma 4.3.

Case 2 $\bar{x}_1 \in D(1)$ and $\bar{x}_2 \in D(2)$. We have:

$$\begin{aligned}
D^*_{\neq} F_1(\bar{x}|\bar{y})(y^*) &= \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ (x,y)\in Gr(F_1) \\ x\neq\bar{x}}} D^* F_1(x|y)(y^*) \\
&= \operatorname*{Limsup}_{\substack{(x_2,y_2)\to(\bar{x}_2,\bar{y}_2) \\ (x_2,y_2)\in Gr(\partial c_2) \\ x_2\neq\bar{x}_2}} D^* F_1(\bar{x}_1, x_2|\bar{y}_1, y_2)(y^*) \\
&\cup \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ (x,y)\in Gr(F_1) \\ x_1\neq\bar{x}_1}} D^* F_1(x_1, \bar{x}_2|y_1, \bar{y}_2)(y^*) \\
&\cup \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ (x,y)\in Gr(F_1) \\ x_1\neq\bar{x}_1, x_2\neq\bar{x}_2}} D^* F_1(x|y)(y^*) \\
&= \operatorname*{Limsup}_{\substack{(x_2,y_2)\to(\bar{x}_2,\bar{y}_2) \\ (x_2,y_2)\in Gr(\partial c_2) \\ x_2\neq\bar{x}_2}} \delta_1(\bar{x}_1, \bar{y}_1, y_1^*) \times \{(c_2)''(x_2) y_2^*\} \\
&\cup \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ (x,y)\in Gr(F_1) \\ x_1\neq\bar{x}_1}} \{(c_1)''(x_1) y_1^*\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*)
\end{aligned}$$

$$\cup \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in Gr(F_1)\\x_1\neq\bar{x}_1,x_2\neq\bar{x}_2}} \{(c_1)''(x_1)y_1^*\} \times \{(c_2)''(x_2)y_2^*\}.$$

Without loss of generality, we suppose that $\bar{x}_1 = a_1^{j_1}$ and $\bar{x}_2 = a_2^{j_2}$ with $j_1 \in \{1, \ldots, q_1\}$ and $j_2 \in \{1, \ldots, q_2\}$.

If $(\bar{y}_1, \bar{y}_2) \in]b_1^{j_1,-}, b_1^{j_1,+}[\times]b_2^{j_2,-}, b_2^{j_2,+}[$, then $\operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in Gr(F_1)\\x\neq\bar{x}}} D^*F_1(x|y)(y^*) = \emptyset$, then the inclusion $D^*_{\neq}F_1(\bar{x}|\bar{y})(y^*) \subset \delta_1(\bar{x}_1, \bar{y}_1, y_1^*) \times \{d_2y_2^*\} \cup \{d_1y_1^*\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*)$ holds.

If $\bar{y}_1 = b_1^{j_1,-}$ and $\bar{y}_2 \in]b_2^{j_2,-}, b_2^{j_2,+}[$, then:

$$\operatorname*{Limsup}_{\substack{(x_2,y_2)\to(\bar{x}_2,\bar{y}_2)\\(x_2,y_2)\in Gr(\partial c_2)\\x_2\neq\bar{x}_2}} \delta_1(\bar{x}_1, \bar{y}_1, y_1^*) \times \{(c_2)''(x_2)y_2^*\} = \emptyset,$$

$$\operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in Gr(F_1)\\x_1\neq\bar{x}_1,x_2\neq\bar{x}_2}} \{(c_1)''(x_1)y_1^*\} \times \{(c_2)''(x_2)y_2^*\} = \emptyset.$$

We can deduce the following equality:

$$D^*_{\neq}F_1(\bar{x}|\bar{y})(y^*) = \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in Gr(F_1)\\x_1\neq\bar{x}_1}} \{(c_1)''(x_1)y_1^*\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*) = \{d_{1j}^- y_1^*\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*).$$

If $\bar{y}_1 = b_1^{j_1,-}$ and $\bar{y}_2 = b_2^{j_2,-}$, then:

$$\operatorname*{Limsup}_{\substack{(x_2,y_2)\to(\bar{x}_2,\bar{y}_2)\\(x_2,y_2)\in Gr(\partial c_2)\\x_2\neq\bar{x}_2}} \delta_1(\bar{x}_1, \bar{y}_1, y_1^*) \times \{(c_2)''(x_2)y_2^*\} = \delta_1(\bar{x}_1, \bar{y}_1, y_1^*) \times \{d_2^{j_2,-} y_2^*\}$$

$$\operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in Gr(F_1)\\x_1\neq\bar{x}_1}} \{(c_1)''(x_1)y_1^*\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*) = \{d_1^{j_1,-} y_1^*\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*)$$

$$\operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\in Gr(F_1)\\x_1\neq\bar{x}_1,x_2\neq\bar{x}_2}} \{(c_1)''(x_1)y_1^*\} \times \{(c_2)''(x_2)y_2^*\} = \{d_1^{j_1,-} y_1^*\} \times \{d_2^{j_2,-} y_2^*\}.$$

We can deduce the following equality:

$$D^*_{\neq}F_1(\bar{x}|\bar{y})(y^*) = \delta_1(\bar{x}_1, \bar{y}_1, y_1^*) \times \{d_2^{j_2,-} y_2^*\} \cup \{d_1^{j_1,-} y_1^*\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*).$$

Given that $d_1^{j_1,-} \geq 0$ and $d_2^{j_2,-} \geq 0$ because c_1 and c_2 are convex functions, we deduce that the inclusion $D^*_{\neq}F_1(\bar{x}|\bar{y})(y^*) \subset \delta_1(\bar{x}_1, \bar{y}_1, y_1^*) \times \{d_2y_2^*\} \cup \{d_1y_1^*\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*)$

always holds with some nonnegative real numbers d_1 and d_2. The other cases can be treated in the same way. □

We now give the result of metric subregularity of Φ.

Theorem 4.2 *Let* $(\bar{x}, \bar{\lambda}) \in \mathbb{R}^2 \times \mathbb{R}^4$ *such that* $0 \in \Phi(\bar{x}, \bar{\lambda})$. *If* $\bar{\lambda}_A > 0$, *then* Φ *is metrically subregular at* $(\bar{x}, \bar{\lambda}, 0)$.

Proof Suppose that we have

$$\begin{array}{l} 0 \in D^*_{\neq} F(\bar{x}| - G(\bar{x}, \bar{\lambda}))(y^*) - Jg_A(\bar{x})^{\perp} z^* \\ y_1^* = 0 \text{ if } \bar{x}_1 = M_1 \\ y_2^* = 0 \text{ if } \bar{x}_2 = M_2 \end{array}$$

with $y^* \in \mathbb{R}^2$ and $z^* \in \mathbb{R}^{|A|}$. By Lemma 4.3, we have

$$\begin{array}{l} 0 \in \delta_1(\bar{x}_1, \bar{y}_1, y_1^*) \times \{d_2 y_2^*\} + \beta \begin{pmatrix} 2y_1^* + y_2^* \\ y_1^* + 2y_2^* \end{pmatrix} - Jg_A(\bar{x})^{\perp} z^* \\ y_1^* = 0 \text{ if } \bar{x}_1 = M_1 \\ y_2^* = 0 \text{ if } \bar{x}_2 = M_2 \end{array}$$

or

$$\begin{array}{l} 0 \in \{d_1 y_1^*\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*) + \beta \begin{pmatrix} 2y_1^* + y_2^* \\ y_1^* + 2y_2^* \end{pmatrix} - Jg_A(\bar{x})^{\perp} z^* \\ y_1^* = 0 \text{ if } \bar{x}_1 = M_1 \\ y_2^* = 0 \text{ if } \bar{x} = M_2 \end{array}$$

with $d_1 \geq 0$ and $d_2 \geq 0$. We consider the following cases:

Case 1: $\bar{x}_1 \in]0, M_1[$ and $\bar{x}_2 \in]0, M_2[$. In this case we have

$$0 \in \{d_1 y_1^*\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*) + \beta \begin{pmatrix} 2y_1^* + y_2^* \\ y_1^* + 2y_2^* \end{pmatrix}$$

or

$$0 \in \delta_1(\bar{x}_1, \bar{y}_1, y_1^*) \times \{d_2 y_2^*\} + \beta \begin{pmatrix} 2y_1^* + y_2^* \\ y_1^* + 2y_2^* \end{pmatrix}.$$

Using the same process as in Proposition 8.2 [14], we can prove that $y^* = 0$.

Case 2: $\bar{x}_1 = M_1$ and $\bar{x}_2 \in]0, M_2[$. In this case we have $A = \{1\}$, then $g_A(x) = x_1 - M_1$, which implies that $Jg_A(x)^{\perp} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The function c_1 is differentiable at M_1 because by (H_3), $D(1) \subset]0, M_1[$. Since $y_1^* = 0$, we have:

$$0 \in \{0\} \times \delta_2(\bar{x}_2, \bar{y}_2, y_2^*) + \beta \begin{pmatrix} y_2^* \\ 2y_2^* \end{pmatrix} - \begin{pmatrix} z^* \\ 0 \end{pmatrix}.$$

By Lemma 4.3, we have $\delta_2(\bar{x}_2, \bar{y}_2, y_2^*) = \{d_2 y_2^*\}$ or $\delta_2(\bar{x}_2, \bar{y}_2, y_2^*) = [d_2 y_2^*, +\infty[$ or $\delta_2(\bar{x}_2, \bar{y}_2, y_2^*) =]-\infty, d_2 y_2^*]$, where $d_2 \geq 0$. Suppose first that

$$\delta_2(\bar{x}_2, \bar{y}_2, y_2^*) = \{d_2 y_2^*\}.$$

In this case we have

$$\begin{pmatrix} \beta & -1 \\ d_2 + 2\beta & 0 \end{pmatrix} \begin{pmatrix} y_2^* \\ z^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of this matrix is equal to $d_2 + 2\beta > 0$, which implies that $(y_2^*, z^*) = (0, 0)$. We suppose now that $\delta_2(\bar{x}_2, \bar{y}_2, y_2^*) = [d_2 y_2^*, +\infty[$. In this case we have:

$$\begin{aligned} &\beta y_2^* - z^* = 0 \\ &(d_2 + 2\beta) y_2^* \leq 0 \\ &y_2^* > 0. \end{aligned}$$

Given that $\beta > 0$ and $y_2^* > 0$, we obtain that $(d_2 + 2\beta) y_2^* > 0$, which contradicts $(d_2 + 2\beta) y_2^* \leq 0$, then the case $\delta_2(\bar{x}_2, \bar{y}_2, y_2^*) = [d_2 y_2^*, +\infty[$ cannot occur. In the same way, we can prove that the case $\delta_2(\bar{x}_2, \bar{y}_2, y_2^*) =]-\infty, d_2 y_2^*]$ cannot occur, finally we obtain that $(y_1^*, y_2^*, z^*) = (0, 0, 0)$.

Case 3: $\bar{x}_1 = M_1$ and $\bar{x}_2 = M_2$. In this case we have $g_A(x) = \begin{pmatrix} x_1 - M_1 \\ x_2 - M_2 \end{pmatrix}$, which implies that $g_A(x)^\perp = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Moreover, c_1 and c_2 are differentiable at M_1 and M_2, respectively, because $D(1) \subset]0, M_1[$ and $D(2) \subset]0, M_2[$, then:

$$\begin{aligned} &\begin{pmatrix} d_1 y_1^* \\ d_2 y_2^* \end{pmatrix} + \beta \begin{pmatrix} 2y_1^* + y_2^* \\ y_1^* + 2y_2^* \end{pmatrix} - \begin{pmatrix} z_1^* \\ z_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &y_1^* = 0 \\ &y_2^* = 0. \end{aligned}$$

This implies that $(y_1^*, y_2^*, z_1^*, z_2^*) = (0, 0, 0, 0)$. Then in all the cases, the set-valued mapping Φ is metrically subregular at $(\bar{x}, \bar{\lambda}, 0)$ by Theorem 4.1. The other cases can be treated in the same way. □

5 Conclusions

In this article, we have obtained a sufficient condition for a generalized equation to be metrically subregular using an object of nonsmooth analysis well adapted to the structure of our problem. We have applied these results to a Cournot duopoly game, extending the metric subregularity result that we obtained in [14]. Two natural extensions of this work would be to obtain the same kind of sufficient conditions for Hölder-metric subregularity in generalized equations and to extend these results in infinite-dimensional spaces.

References

1. Chuong, T.D., Kruger, A.Y., Yao, J.C.: Calmness of efficient solution maps in parametric vector optimization. J. Global Optim. **51**(4), 677–688 (2011)
2. Dontchev, A.L., Rockafellar, R.T.: Implicit Functions and Solution Mappings. Springer, Dordrecht (2009)
3. Fabian, M.J., Henrion, R., Kruger, A.Y., Outrata, J.V.: Error bounds: necessary and sufficient conditions. Set-Valued Var. Anal. **18**(2), 121–149 (2010)
4. Ioffe, A.D., Outrata, J.V.: On metric and calmness qualification conditions in subdifferential calculus. Set-Valued Anal. **16**(2–3), 199–227 (2008)
5. Kruger, A.Y.: Error bounds and metric subregularity. Optimization **64**(1), 49–79 (2014)
6. Kruger, A.Y.: Error bounds and Holder metric subregularity. Set-Valued Var. Anal. **23**(4), 705–736 (2015)
7. Zheng, X.Y., Ng, K.F.: Metric subregularity and calmness for nonconvex generalized equations in Banach spaces. SIAM J. Optim. **20**(5), 2119–2136 (2005)
8. Zheng, X.Y., Ng, K.F.: Metric subregularity for proximal generalized equations in Hilbert spaces. Nonlinear Anal. **75**(3), 1686–1699 (2005)
9. Mordukhovich, B.S., Ouyang, W.: Higher-orden metric subregularity and its applications. J. Global Optim. **63**(4), 777–795 (2015)
10. Henrion, R., Surowiec, T.: On calmness conditions in convex bilevel programming. Appl. Anal. **90**(6), 951–970 (2011)
11. Henrion, R., Outrata, J., Surowiec, T.: Analysis of M-stationary points to an EPEC modeling oligopolistic competition in an electricity spot market. ESAIM: Control Optim. Calc. Var. **18**(2), 295–317 (2012)
12. Henrion, R., Kruger, A.Y., Outrata, J.V.: Some remarks on stability of generalized equations. J. Optim. Theory Appl. **159**(3), 681–697 (2013)
13. Adly, S., Cibulka, R.: Quantitative stability of a Generalized Equation. J. Optim. Theory Appl. **160**, 90–110 (2014)
14. Maréchal, M., Correa, R.: Error bounds, metric subregularity and stability in generalized nash equilibrium problems with nonsmooth payoff functions. Optimization **65**(10), 1829–1854 (2016)
15. Gfrerer, H.: On directional metric subregularity and second-order optimality conditions for a class of nonsmooth mathematical programs. SIAM J. Optim. **23**(1), 632–665 (2013)
16. Gfrerer, H.: On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs. Set-Valued Var. Anal. **21**(2), 151–176 (2013)
17. Gfrerer, H., Klatte, D.: Lipschitz and Hlder stability of optimization problems and generalized equations. Math. Program. **158**(1–2), 35–75 (2016)
18. Mordukhovich, B.S.: Stability theory for parametric generalized equations and variational inequalities via nonsmooth analysis. Trans. Am. Math. Soc. **343**(2), 609–657 (1994)
19. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation I. Basic Theory. Springer, Berlin (2006)
20. Robinson, S.M.: Some continuity properties of polyhedral multifunctions. Math. Program. Study **14**, 206–214 (1981)