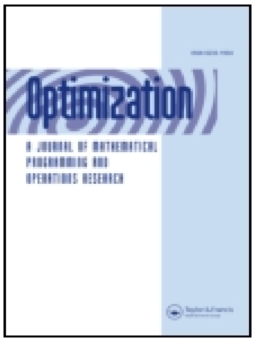

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M. Maréchal & R. Correa

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Error bounds, metric subregularity and stability in Generalized Nash Equilibrium Problems with nonsmooth payoff functions

M. Maréchal[a] and R. Correa[b]

[a]Facultad de Ingeniería, Universidad Diego Portales, Santiago, Chile; [b]Rectory, Universidad de O'Higgins and DIM, CMM Universidad de Chile, Santiago, Chile

ABSTRACT
In this paper, we study the calmness of a generalized Nash equilibrium problem (GNEP) with non-differentiable data. The approach consists in obtaining some error bound property for the KKT system associated with the generalized Nash equilibrium problem, and returning to the primal problem thanks to the Slater constraint qualification.

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1. Introduction

This paper deals with error bounds, metric regularity, metric subregularity and stability in the generalized Nash equilibrium problem (GNEP) with non-smooth pay-off functions. Generalized Nash equilibrium problems have many applications in economy and other areas (see e.g. [1] for examples of applications). Many authors [2–4] have studied the error bound property for the KKT system associated with GNEP when the pay-off functions for all players are differentiable; other authors [5,6] have studied the error bound property for complementarity systems. In these articles, the error bound property is used in order to derive some convergence results of an LP–Newton algorithm which is a very well-adapted algorithm to solve equations with non-isolated solutions. Rockafellar has written about a variational approach to stability in the Nash equilibrium problem based on monotonicity assumptions.[7] For this approach, it is necessary to have strong assumptions of regularity about the pay-off function of all players.

In this paper, we provide results of metric (sub)regularity and error bounds about GNEPs when the loss functions are not differentiable, under the assumption of strict complementarity. We apply these results to the stability of perturbated GNEPs and to a special case of two-player game (which is larger than Cournot duopoly games).

The article is organized as follows: Section 2 introduces the notation and the main assumptions about GNEP. Section 3 gives the definition of the coderivative introduced by Mordukhovich [8–10], and the results useful for our quantitative study of GNEP. Section 4 gives some results about metric regularity and metric subregularity under the assumption of strict complementarity for the KKT system associated with GNEP and Section 5 gives the main results about stability of GNEP with respect to a parameter. Finally, Section 6 proposes the application of the results obtained in the

CONTACT M. Maréchal matthieu.marechal@mail.udp.cl

previous sections to a class of two-player games which is larger than the Cournot duopoly games (see e.g. [11–14]).

1.1. Some notations

Throughout the paper, $\|\cdot\|$ denotes the Euclidian norm on $\mathbb{R}^q$, with $q \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, and $\langle \cdot, \cdot \rangle$ is the classical inner product on $\mathbb{R}^q$ associated with the norm $\|\cdot\|$. Let (U, δ) be a metric space, for $r > 0$ and a given point $x \in U$, we set $B(x, r) := \{z \in U \;:\; \delta(x, z) < r\}$. The distance from a point $x \in \mathbb{R}^q$ to a set $B \subset \mathbb{R}^q$ is denoted by $\mathrm{dist}(x, B)$ and defined by $\mathrm{dist}(x, B) := \inf_{z \in B} \|x - z\|$. Let $q_1, q_2 \in \mathbb{N}^*$; if $f : \mathbb{R}^{q_1} \to \mathbb{R}^{q_2}$ is a differentiable function at a point $x \in \mathbb{R}^{q_1}$, then its Jacobian matrix at this point is denoted by $J_f(x)$. Let A be an arbitrary matrix, $A^\top$ stands for its transpose. When B is a finite set, then $|B|$ denotes its cardinality. For an arbitrary set B, $B^{\mathbb{N}}$ denotes all the sequences with elements in B.

2. The Generalized Nash Equilibrium Problem

The GNEP is a Nash game in which each player's strategy depends on the other players' strategies. More precisely, assume that there are p players and each player ν controls the variable $x^\nu \in \mathbb{R}^{n^\nu}$ as a strategy. Let us denote by x the vector of strategies

$$x := (x^1, \ldots, x^p) \quad \text{and} \quad n := n_1 + n_2 + \cdots + n_p.$$

Let us denote by $x^{-\nu}$ the vector formed by all players' decision variables except player ν. So we can also write $x = (x^\nu, x^{-\nu})$, which is a shortcut (already used in many papers on the subject; see e.g. [15,16]) to denote the vector $x = (x^1, \ldots, x^{\nu-1}, x^\nu, x^{\nu+1}, \ldots, x^p)$. We define the set-valued mapping $X^\nu : \mathbb{R}^{n^{-\nu}} \rightrightarrows \mathbb{R}^{n^\nu}$, where $n^{-\nu} = n - n^\nu$, such that the strategy of player ν belongs to the set $X^\nu(x^{-\nu})$. The aim of player ν is, given the strategy $x^{-\nu}$, to choose a strategy x^ν such that x^ν solves the following optimization problem

$$(P^\nu) \quad \min_{x^\nu} \theta^\nu(x^\nu, x^{-\nu}), \quad \text{subject to} \quad x^\nu \in X^\nu(x^{-\nu}),$$

where $\theta^\nu(x^\nu, x^{-\nu})$ denotes the loss that player ν suffers when the rival players have chosen the strategy $x^{-\nu}$. The GNEP consists in finding $\bar{x} \in \mathbb{R}^n$ such that for all $\nu \in \{1, \ldots, p\}$:

$$\bar{x}^\nu \in \arg \min_{X^\nu(\bar{x}^{-\nu})} \theta^\nu(\cdot, \bar{x}^{-\nu}).$$

We make the following assumptions:

(1) The set $X^\nu(x^{-\nu})$ is defined by $X^\nu(x^{-\nu}) = \{x^\nu \in \mathbb{R}^{n^\nu} \;:\; g^\nu(x^\nu, x^{-\nu}) \leq 0\}$, with $g^\nu : \mathbb{R}^n \to \mathbb{R}^{m^\nu}$ a C^2 function such that for all $x^{-\nu} \in \mathbb{R}^{n^{-\nu}}$, $g^\nu(\cdot, x^{-\nu})$ is convex. This assumption implies that each player ν controlling the variable x^ν solves the following problem

$$\min_{g^\nu(x^\nu, x^{-\nu}) \leq 0} \theta^\nu(x^\nu, x^{-\nu}).$$

(2) For each $x^{-\nu} \in \mathbb{R}^{n^{-\nu}}$, we have

$$N(x^\nu, X^\nu(x^{-\nu})) = \{J_{x^\nu} g^\nu(x^\nu, x^{-\nu})^\top \xi \mid \xi \in N(g^\nu(x^\nu, x^{-\nu}), (\mathbb{R}_-)^{m^\nu})\}$$

where when K is a closed convex set in $\mathbb{R}^q$, $q \in \mathbb{N}^*$, and $x \in K$,

$$N(x, K) = \{x^* \in \mathbb{R}^q \mid \forall x' \in K \,,\, \langle x^*, x' - x \rangle \leq 0\} \tag{1}$$

This assumption ensures the existence of Lagrange multipliers at any solution for player ν and it is implied by the standard qualification conditions like LICQ and MFCQ.

(3) The functions θ^ν are not necessarily differentiable, but we suppose that for all $x^{-\nu} \in \mathbb{R}^{n^{-\nu}}$, $\theta^\nu(\cdot, x^{-\nu})$ is convex on $\mathbb{R}^{n^\nu}$ and θ^ν is locally Lipschitz-continuous on $\mathbb{R}^n$.

Such a game will be denoted by $\mathcal{G} := ((\theta^\nu)_{\nu=1,\dots,p}, (g^\nu)_{\nu=1,\dots,p})$. This notation will be used in Proposition 5.4.

For every player $\nu \in \{1, \dots, p\}$, we introduce the Lagrangian function $L^\nu : \mathbb{R}^{n^\nu} \times \mathbb{R}^{n^{-\nu}} \times \mathbb{R}^{m^\nu} \to \mathbb{R}$ which is given by

$$L^\nu(x^\nu, x^{-\nu}, \lambda^\nu) := \theta^\nu(x^\nu, x^{-\nu}) + \langle \lambda^\nu, g^\nu(x^\nu, x^{-\nu})\rangle.$$

Therefore, $\bar{x} := (\bar{x}^1, \dots, \bar{x}^p)$ is a generalized Nash equilibrium if and only if there exists a vector $\bar{\lambda} := (\bar{\lambda}^1, \dots, \bar{\lambda}^p)$ such that for all $\nu \in \{1, \dots, p\}$, $(\bar{x}^\nu, \bar{\lambda}^\nu)$ is a solution of the KKT system

$$0 \in \partial_{x^\nu} L^\nu(\bar{x}^\nu, \bar{x}^{-\nu}, \bar{\lambda}^\nu) \text{ and } 0 \leq \bar{\lambda}^\nu \perp (-g^\nu(\bar{x}^\nu, \bar{x}^{-\nu})) \geq 0 \tag{2}$$

where $\partial_{x^\nu} L^\nu(\bar{x}^\nu, \bar{x}^{-\nu}, \bar{\lambda}^\nu)$ stands for the convex subdifferential of the function L^ν with respect to the variable x^ν. Recall that for a convex function $f : \mathbb{R}^q \to \mathbb{R}$, the convex subdifferential of f at $\bar{x}$ is defined by

$$\partial f(\bar{x}) := \left\{x^* \in \mathbb{R}^q \mid \forall x \in \mathbb{R}^q\ , \ \langle x^*, x - \bar{x}\rangle \leq f(x) - f(\bar{x})\right\}.$$

For two vectors $a, b \in \mathbb{R}^q$, with $q \in \mathbb{N}^*$, one has $0 \leq a \perp b \geq 0$ if and only if $\min\{a, b\} = 0$, where $\min\{a, b\} := (\min\{a_i, b_i\})_{i=1,\dots,q}$. Therefore, the above system becomes

$$0 \in \partial_{x^\nu} L(\bar{x}^\nu, \bar{x}^{-\nu}, \bar{\lambda}^\nu) \text{ and } \min\{\bar{\lambda}^\nu, -g^\nu(\bar{x}^\nu, \bar{x}^{-\nu})\} = 0. \tag{3}$$

We introduce the set-valued mapping $\Phi : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ defined by

$$\forall z := (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m\ , \ \Phi(z) := \begin{pmatrix} T(z) \\ \min\{\lambda^1, -g^1(x^1, x^{-1})\} \\ \vdots \\ \min\{\lambda^p, -g^p(x^p, x^{-p})\} \end{pmatrix} \tag{4}$$

where

$$T(z) := \partial_{x^1} L^1(x^1, x^{-1}, \lambda^1) \times \cdots \times \partial_{x^p} L^p(x^p, x^{-p}, \lambda^p) \text{ and } m := \sum_{\nu=1}^{p} m^\nu. \tag{5}$$

A vector $\bar{x} := (\bar{x}^1, \dots, \bar{x}^p) \in \mathbb{R}^n$ is a generalized Nash equilibrium if and only if there exists a vector $\bar{\lambda} := (\bar{\lambda}^1, \dots, \bar{\lambda}^p) \in \mathbb{R}^m$ satisfying

$$0 \in \Phi(\bar{z}) \text{ with } \bar{z} := (\bar{x}, \bar{\lambda}). \tag{6}$$

We denote by Ω the set of solutions of the generalized system (6), that is

$$\Omega := \{z \in \mathbb{R}^{n+m}\ :\ 0 \in \Phi(z)\}.$$

The aim of this paper is to obtain the following local error bound property for an element $\bar{z} \in \Omega$

$$\exists r, L > 0, \quad \forall z \in B(\bar{z}, r),\ \operatorname{dist}(z, \Omega) \leq L \operatorname{dist}(0, \Phi(z)), \tag{7}$$

and to use this error bound property in order to obtain some results of stability of the solution map of the GNEP with respect to a parameter. Since the error bound property (7) is very closely related to the metric subregularity of Φ at $(\bar{z}, 0)$ (Definition 1), Section 4 studies the metric regularity and subregularity of Φ.

Since we don't have differentiability, we introduce in the next section the coderivative, developed by Boris Mordukhovich, and we will recall some important properties satisfied by the coderivative.

The function $(a, b) \mapsto \min\{a, b\}$ is called a NCP-function, which characterizes that $\min\{a, b\} = 0 \Leftrightarrow 0 \leq a \perp b \geq 0$. There exist other NCP-functions, for example, the very classical Fischer–Burmeister function which is defined by $(a, b) \mapsto a + b - \sqrt{a^2 + b^2}$. The Fischer–Burmeister function is strongly semi-smooth and has a continuously differentiable square; then, it is superior to the NCP-function we use in this article, but for our purpose, the NCP-function we use provides the same results as the Fischer–Burmeister function and the calculus are less complicated.

3. Coderivative and calmness

This section is directly inspired by the article of Mordukhovich [9] and his book.[10] We introduce some notations: consider a set-valued mapping $T : X \rightrightarrows Y$, where X and Y are two Euclidian spaces. We define the domain of T by $\mathrm{dom}(T) := \{x \in X \;:\; T(x) \neq \emptyset\}$, the graph of T by $\mathrm{Gr}(T) := \{(x, y) \in X \times Y \;:\; y \in T(x)\}$ and the inverse map of T by $T^{-1}(y) := \{x \in X \;:\; y \in T(x)\}$ for all $y \in Y$.

Let $\bar{x} \in \mathrm{dom}(T)$. The limsup of T at $\bar{x}$ is given by:

$$\operatorname*{Limsup}_{x \to \bar{x}} T(x) := \left\{ \lim_{n \to +\infty} x_n^* \;\middle|\; \begin{array}{l} (x_n^*)_n \text{ converges and for all } n \in \mathbb{N}, \\ \text{there exists } x_n \in X \text{ with } x_n \to \bar{x} \text{ and } x_n^* \in T(x_n) \end{array} \right\}.$$

Let $K \subset X$ be a closed subset in X and $x \in K$, we define the Fréchet normal cone, also known as the regular normal cone, by:

$$\hat{N}(x, K) := \left\{ x^* \in X \;:\; \limsup_{x' \xrightarrow{K} \bar{x}} \frac{\langle x^*, x' - x \rangle}{\|x' - x\|} \leq 0 \right\}.$$

The limiting normal cone, also known as the normal cone of Mordukhovich, is defined by:

$$N_L(\bar{x}, K) := \operatorname*{Limsup}_{x \xrightarrow{K} \bar{x}} \hat{N}(x, K).$$

If K is a convex set, then $N_L(\bar{x}, K) = N(\bar{x}, K)$, where $N(\bar{x}, K)$ has been defined in (1).

Let $T : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \mathrm{Gr}(T)$. The coderivative $D^*T(\bar{x}|\bar{y}) : Y \rightrightarrows X$ is given by:

$$\forall y^* \in Y,\; D^*T(\bar{x}|\bar{y})(y^*) := \{x^* \in X : (x^*, -y^*) \in N_L((\bar{x}, \bar{y}), \mathrm{Gr}(T))\}$$

The coderivative is related to the Jacobian of a strictly differentiable mapping, as the following proposition shows us. For a single-valued map f, the notation $D^*f(x)(y^*)$ means $D^*f(x|f(x))(y^*)$.

Proposition 3.1: [9, Proposition 2.5] *Let $f : \mathbb{R}^p \to \mathbb{R}^q$, with $p, q \in \mathbb{N}^*$, be a strictly differentiable single-valued mapping $\bar{x} \in \mathbb{R}^p$. Then, for all $\xi \in \mathbb{R}^q$, we have*

$$D^*f(\bar{x})(\xi) = Jf(\bar{x})^\top \xi,$$

The following theorem gives a very important calculus rule for the coderivative.

Theorem 3.2: [10, Theorem 1.62] *Let $f : \mathbb{R}^p \to \mathbb{R}^q$, with $p, q \in \mathbb{N}^*$, be a strictly differentiable single-valued mapping at $\overline{x} \in \mathbb{R}^p$ and let $T : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ be a multifunction. We then have:*

$$\forall \xi \in \mathbb{R}^q,\ D^*(T+f)(\overline{x}|\overline{y}+f(\overline{x}))(\xi) = D^*T(\overline{x}|\overline{y})(\xi) + Jf(\overline{x})^\top \xi. \tag{8}$$

The following defines the metric subregularity.

Definition 1: Let $T : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ with $p, q \in \mathbb{N}^*$, and $(\bar{x}, \bar{y}) \in \mathrm{Gr}(T)$. We say that T is metrically subregular at $(\bar{x}, \bar{y}) \in \mathrm{Gr}(T)$ iff there exist constants $r > 0$, $L \geq 0$ such that:

$$\forall x \in B(\overline{x}, r),\ \mathrm{dist}(x, T^{-1}(\bar{y})) \leq L\mathrm{dist}(\bar{y}, T(x))$$

Metric subregularity is closely related to the error bound property in the following sense: consider $\bar{z} \in \Omega$, where Ω is the set of solutions of the equation (6), which means that $0 \in \Phi(\bar{z})$, where Φ has been defined in (4). The set-valued mapping Φ is metrically subregular at $(\bar{z}, 0)$ if and only if there exist constants $r > 0$, $L \geq 0$ such that:

$$\forall z \in B(\bar{z}, r),\ \mathrm{dist}(z, \Phi^{-1}(0)) \leq L\mathrm{dist}(0, \Phi(z))$$

Since $\Phi^{-1}(0) = \Omega$, Φ is metrically subregular at $(\bar{z}, 0)$ if and only if the following local error bound property

$$\forall z \in B(\bar{z}, r),\ \mathrm{dist}(z, \Omega) \leq L\mathrm{dist}(0, \Phi(z))$$

holds on.

The coderivative gives a criterion for metric regularity which is a stronger property than metric subregularity. In order to use the full power of this criterion, in the next section, we obtain some sufficient conditions for metric Regularity; then, we deduce the metric subregularity and an error bound property. The following defines the metric regularity.

Definition 2: Let $T : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ with $p, q \in \mathbb{N}^*$, and $(\bar{x}, \bar{y}) \in \mathrm{Gr}(T)$. We say that T is metrically regular near $(\bar{x}, \bar{y}) \in \mathrm{Gr}(T)$ iff there exist constants $r > 0$, $L \geq 0$ such that:

$$\forall y \in B(\bar{y}, r),\ \forall x \in B(\overline{x}, r),\ \mathrm{dist}(x, T^{-1}(y)) \leq L\mathrm{dist}(y, T(x))$$

Taking $y = \bar{y}$ in Definition 2, we easily verify that metric regularity implies metric subregularity.

The coderivative criterion for metric regularity requires that the set-valued mapping has a local closed graph at the reference point. The following lemma shows that the set-valued mapping Φ has a closed graph.

Lemma 3.3: *The set-valued mapping Φ has a closed graph.*

Proof: Let $(z_n, d_n)_n \in (\mathrm{Gr}(\Phi))^{\mathbb{N}}$ be a convergence sequence to a limit $(\bar{z}, \overline{d})$. We show that $\overline{d} \in \Phi(\bar{z})$. We can write $d_n = (d_n^1, d_n^2)$, with $d_n^1 \in T(z_n)$ and

$$d_n^2 = \begin{pmatrix} \min\{\lambda_n^1, -g^1(x_n^1, x_n^{-1})\} \\ \vdots \\ \min\{\lambda_n^p, -g^p(x_n^p, x_n^{-p})\} \end{pmatrix}.$$

By continuity of g, we have

$$\overline{d}^2 = \begin{pmatrix} \min\{\bar{\lambda}^1, -g^1(\overline{x}^1, \overline{x}^{-1})\} \\ \vdots \\ \min\{\bar{\lambda}^p, -g^p(\overline{x}^p, \overline{x}^{-p})\} \end{pmatrix}.$$

Writing $d_n^1 = (d_n^{1,\nu})_{\nu=1,\dots,p}$, with $d_n^{1,\nu} \in \mathbb{R}^{n^\nu}$, we have, for each player ν

$$d_n^{1,\nu} \in \partial_{x^\nu}\theta^\nu(x_n^\nu, x_n^{-\nu}) + J_{x^\nu}g^\nu(x_n^\nu, x_n^{-\nu})^\top\lambda_n^\nu,$$

thus

$$d_n^{1,\nu} - J_{x^\nu}g^\nu(x_n^\nu, x_n^{-\nu})^\top\lambda_n^\nu \in \partial_{x^\nu}\theta^\nu(x_n^\nu, x_n^{-\nu}).$$

In order to simplify the notation, let us denote

$$\tilde{d}_n^{1,\nu} = d_n^{1,\nu} - J_{x^\nu}g^\nu(x_n^\nu, x_n^{-\nu})^\top\lambda_n^\nu.$$

For all $y^\nu \in \mathbb{R}^{n^\nu}$, for all $n \in \mathbb{N}$, we have

$$\langle \tilde{d}_n^{1,\nu}, y^\nu - x_n^\nu\rangle \leq \theta^\nu(y^\nu, x_n^{-\nu}) - \theta^\nu(x_n^\nu, x_n^{-\nu}).$$

Passing to the limit when $n \to +\infty$, and denoting

$$\tilde{d}^{1,\nu} = \overline{d}^{1,\nu} - J_{\overline{x}^\nu}g^\nu(\overline{x}^\nu, \overline{x}^{-\nu})^\top\bar{\lambda}^\nu,$$

we obtain that

$$\tilde{d}^{1,\nu} \in \partial_{x^\nu}\theta^\nu(\overline{x}^\nu, \overline{x}^{-\nu}).$$

Finally, we have $\overline{d}^1 \in T(\bar{z})$, which permits us to conclude that $\overline{d} \in \Phi(\bar{z})$, thus Φ has a closed graph. □

It is the local closedness of the graph of Φ at the reference point that is important for the application of the coderivative criterion for metric regularity due to Mordukhovich. Since it is a direct application of a result in [9], we do not give the proof.

Theorem 3.4: [9, Corollary 3.8] *Let $\bar{z} \in \Omega$. The set-valued mapping Φ defined by (4) is metrically regular around $(\bar{z}, 0)$ if and only if for all $\xi \in \mathbb{R}^{n+m}$, $0 \in D^*\Phi(\bar{z}|0)(\xi)$ implies $\xi = 0$.*

In Section 5, we study the stability of the GNEP with respect to a parameter in terms of calmness, which means we give some sufficient condition for the solution map of GNEP depending on a parameter to be calm. The calmness is defined as follows.

Definition 3: Let $T : U \rightrightarrows \mathbb{R}^q$ with $q \in \mathbb{N}^*$ and (U, δ) a metric space. Let $(\bar{x}, \bar{y}) \in \mathrm{Gr}(T)$. We say that T is calm at $(\bar{x}, \bar{y})$ iff there exist $r, L > 0$ such that for all $x \in B(\bar{x}, r)$, we have

$$T(x) \cap B(\bar{y}, r) \subset T(\overline{x}) + B(0, L\delta(x, \overline{x})).$$

It is well known that the metric subregularity of a set-valued mapping T at $(\bar{x}, \bar{y}) \in \mathrm{Gr}(T)$ is equivalent to the calmness of T^{-1} at $(\bar{y}, \bar{x})$ (see e.g. [17, Theorem 3H.3]). In the same way, the metric regularity of a set-valued mapping T at $(\bar{x}, \bar{y}) \in \mathrm{Gr}(T)$ is equivalent to the Aubin property of T^{-1} at $(\bar{y}, \bar{x})$ (the Aubin property, which is also known as Lipschitz-like and pseudo-Lipschitz, is not defined here because we do not use this notion in this paper; for the proof of equivalence, see e.g. [17, Theorem 3E6]). We precise that the Aubin property is characterized by the Mordukhovich criterion (see e.g. [10]).

4. Metric regularity properties for the Generalized Nash Equilibrium Problem under assumption of strict complementarity

In this section, we provide some results about the metric regularity and the metric subregularity for the set-valued mapping Φ, and deduce an error bound property for the GNEP. The proofs follow the path of the articles of Facchinei et al. (see e.g. [2,3]) and the article of Izmailov and Solodov [4]. First, we define some sets of indices.

We define the two sets of indices $\mathcal{I} := \{1, \ldots, m\}$, with $m := \sum_{\nu=1}^{p} m_\nu$, and

$$\mathcal{J} := \{(\nu, j) \ : \ \nu \in \{1, \ldots, p\}\,, \, j \in \{1, \ldots, m^\nu\}\}.$$

Let us define the mapping $\varphi : \mathcal{J} \to \mathcal{I}$ by

$$\forall \nu \in \{1, \ldots, p\},\ \forall j \in \{1, \ldots, m^\nu\},\ \varphi(\nu, j) := \sum_{\mu<\nu} m^\mu + j \tag{9}$$

with the convention

$$\sum_{\mu<1} m^\mu := 0.$$

We observe that φ is a bijection from $\mathcal{J}$ to $\mathcal{I}$, and it is strictly increasing for the lexicographic order, that is for all $(\nu_1, j_1) \in \mathcal{J}$, for all $(\nu_2, j_2) \in \mathcal{J}$,

$$\nu_1 < \nu_2 \text{ or } [\nu_1 = \nu_2 \text{ and } j_1 < j_2] \Longrightarrow \varphi(\nu_1, j_1) < \varphi(\nu_2, j_2).$$

Therefore, we can define the family of functions $(\tilde{g}_i)_{i\in\mathcal{I}}$ with $\tilde{g}_i : \mathbb{R}^n \to \mathbb{R}$ satisfying:

$$\forall i \in \mathcal{I},\ \tilde{g}_i := g_j^\nu \text{ where } (\nu, j) := \varphi^{-1}(i). \tag{10}$$

We can observe that equality (10) implies that for all $(\nu, j) \in \mathcal{J}$, $g_j^\nu = \tilde{g}_{\varphi(\nu,j)}$.

This bijection allows us to write $(g^\nu)_{\nu=1,\ldots,p} = (\tilde{g}_i)_{i\in\mathcal{I}}$.

Example 4.1: Consider the following Nash game:

Player 1:

$$\min_{x_1} \theta^1(x^1, x^2) \text{ s. t. } g_1^1(x) \le 0,\ g_2^1(x) \le 0$$

and

Player 2:

$$\min_{x_2} \theta^2(x^1, x^2) \text{ s. t. } g_1^2(x) \le 0,\ g_2^2(x) \le 0,\ g_3^2(x) \le 0$$

We have $\mathcal{I} = \{1, 2, 3, 4, 5\}$, $\mathcal{J} = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3)\}$ and, since $m^1 = 2$, we have:

$$\varphi(1, 1) = 1,\ \varphi(1, 2) = 2,\ \varphi(2, 1) = 3,\ \varphi(2, 2) = 4,\ \varphi(2, 3) = 5.$$

In this case:

$$\tilde{g}_1(x) = g_1^1(x),\ \tilde{g}_2(x) = g_2^1(x),\ \tilde{g}_3(x) = g_1^2(x),\ \tilde{g}_4(x) = g_2^2(x),\ \tilde{g}_5(x) = g_3^2(x).$$

Let us define the sets of active indices, at $x \in \mathbb{R}^n$,

$$A(x) := \{i \in \mathcal{I} \ : \ \tilde{g}_i(x) = 0\} \tag{11}$$

and, for all $\nu \in \{1, \ldots, p\}$,

$$A^\nu(x) := \{j \in \{1, \ldots, m^\nu\} \ : \ g_j^\nu(x) = 0\} = \{j \in \{1, \ldots, m^\nu\} \ : \ \varphi(\nu, j) \in A(x)\}.$$

For every set of indices $\beta \subset \{1, \ldots, m\}$ and every player ν, we define the set β^ν of indices corresponding to ν which contribute to β:

$$\beta^\nu := \{j \in \{1, \ldots, m_\nu\} \ : \ \varphi(\nu, j) \in \beta\} \tag{12}$$

with φ defined in (9).

We introduce the notation $C(A(x)) := \mathcal{I} \setminus A(x)$, that is

$$C(A(x)) = \{i \in \mathcal{I} \ : \ \tilde{g}_i(x) \neq 0\}. \tag{13}$$

In what follows, when we fix an element $\bar{z} = (\bar{x}, \bar{\lambda})$, we use the notations $A := A(\bar{x})$, $C(A) := C(A(\bar{x}))$ and $A^\nu := A^\nu(\bar{x})$. Moreover, for any vector $Z \in \mathbb{R}^q$ with $q \in \mathbb{N}^*$ and for any set of indices $J \subset \{1, \ldots, q\}$, the notation Z_J is defined as follows:

$$Z_J := (Z_j)_{j \in J}. \tag{14}$$

The notation $Z_J > 0$ means $Z_j > 0$, for all $j \in J$.

We say that the strict complementarity assumption holds at $\bar{z} \in \Omega$ if for all $i \in \mathcal{I}$, $\bar{\lambda}_i > 0$ or $\tilde{g}_i(\bar{x}) < 0$. Therefore, the function $z \to (\min\{\lambda_i, -\tilde{g}_i(x)\})_{i \in \mathcal{I}}$ is C^2 in a neighbourhood of $\bar{z}$, so it is strictly differentiable.

Let us define the functions $\Phi_1 : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+m}$ and $\Phi_2 : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ by

$$\Phi_1(z) := F(x) \times \{0^m\}, \quad \Phi_2(z) := \begin{pmatrix} G(x, \lambda) \\ \min\{\lambda^1, -g^1(x)\} \\ \vdots \\ \min\{\lambda^p, -g^p(x)\} \end{pmatrix} \tag{15}$$

where $\min\{a, b\} := (\min\{a_i, b_i\})_{i=1,\ldots,q}$ if $a, b \in \mathbb{R}^q$ and F, G is defined as follows:

$$F(x) := \partial_{x^1}\theta^1(x^1, x^{-1}) \times \cdots \times \partial_{x^p}\theta^p(x^p, x^{-p}) \tag{16}$$

and

$$G(x, \lambda) := \begin{pmatrix} J_{x^1} g^1(x^1, x^{-1})^\top \lambda^1 \\ \vdots \\ J_{x^p} g^p(x^p, x^{-p})^\top \lambda^p \end{pmatrix} \tag{17}$$

Observe that $\Phi(z) = \Phi_1(z) + \Phi_2(z)$. Under the strict complementarity assumption, the function Φ_2 is strictly differentiable at $\bar{z}$; therefore,

$$D^*\Phi(\bar{z}|0)(\xi) = D^*\Phi_1(\bar{z}| - \Phi_2(\bar{z}))(\xi) + J\Phi_2(\bar{z})^\top \xi. \tag{18}$$

Moreover, we can observe that

$$\mathrm{Gr}(\Phi_1) = \{(x, \lambda, y, 0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \ : \ (x, y) \in \mathrm{Gr}(F)\}.$$

Therefore, for all $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ and $y \in F(x)$, we have:

$$N_L((x, \lambda, y, 0), \mathrm{Gr}(\Phi_1)) = \{(x^*, 0, \xi_1, \xi_2) | (x_*, \xi_1) \in N_L((x, y), \mathrm{Gr}(F)) \ , \ \xi_2 \in \mathbb{R}^m\}$$

which implies that for any $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ and $y \in F(x)$, we have, for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$D^*\Phi_1((x, \lambda)|(y, 0))(\xi) = D^*F(x|y)(\xi_1) \times \{0_m\}. \tag{19}$$

Before giving the first result about the existence of an error bound for the GNEP, we provide a technical lemma.

Lemma 4.2: *Let $\bar{z} \in \Omega$ and $\beta \subset A$ such that $\bar{\lambda}_\beta > 0$, where the meaning of $\bar{\lambda}_\beta$, when β is a set, is given by (14).*

Define the set-valued mapping $\Phi_\beta : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+|\beta|+|C(A)|}$ by

$$\Phi_\beta(z) := \begin{pmatrix} T(z) \\ (\min\{\lambda_i, -\tilde{g}_i(x)\})_{i\in\beta\cup C(A)} \end{pmatrix}.$$

We consider the set $\Omega_\beta := \{z \in \mathbb{R}^{n+m} \ : \ 0 \in \Phi_\beta(z)\}$. We assume that the following condition holds

$$\left.\begin{array}{l} 0 \in D^*F(\bar{x}| - G(\bar{x},\bar{\lambda}))(\xi_1) + J_xG(\bar{x},\bar{\lambda})^\top\xi_1 - J\tilde{g}_\beta(\bar{x})^\top\xi_2 \\ \forall \nu \in \{1,\dots,p\}\ ,\ J_{x^\nu}g^\nu_{\beta^\nu}(\bar{x})\xi_1^\nu = 0 \ \text{if}\ \beta^\nu \neq \emptyset \end{array}\right\} \Rightarrow \xi = 0, \tag{20}$$

with β^ν defined in (12), $\xi_1 = (\xi_1^\nu)_{\nu=1,\dots,p}$ and $\xi_1^\nu \in \mathbb{R}^{n_\nu}$.

Then, the set-valued mapping Φ_β is metrically regular near $(\bar{z}, 0)$.

Proof: Let $\xi := (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^n \times \mathbb{R}^{|\beta|} \times \mathbb{R}^{|C(A)|}$. We show that $0 \in D^*\Phi_\beta(\bar{z}|0)(\xi)$ implies $\xi = 0$. We first observe that $\bar{\lambda}_\beta > 0$ implies the existence of a real $r > 0$ such that after some permutations of the coordinates of $(\tilde{g}_i)_{i\in\mathcal{I}}$, for all $z := (x,\lambda) \in B(\bar{z}, r)$,

$$\Phi_\beta(z) = \begin{pmatrix} T(x,\lambda) \\ -\tilde{g}_\beta(x) \\ \lambda_{C(A)} \end{pmatrix} = \Phi_1(z) + \begin{pmatrix} G(x,\lambda) \\ -\tilde{g}_\beta(x) \\ \lambda_{C(A)} \end{pmatrix}$$

where Φ_1 has been defined in (15). We introduce $\Phi_{2,\beta}(z) := \begin{pmatrix} G(x,\lambda) \\ -\tilde{g}_\beta(x) \\ \lambda_{C(A)} \end{pmatrix}$.

Therefore, by the above equality, (19) and by [10, Theorem 1.62], we obtain the following estimations of $D^*\Phi_\beta(\bar{z}|0)(\xi)$:

$$\begin{aligned} D^*\Phi_\beta(\bar{z}|0)(\xi) &= D^*\Phi_1(\bar{z}| - \Phi_{2,\beta}(\bar{z}))(\xi) + J\Phi_{2,\beta}(\bar{z})^\top\xi \\ &= D^*F(\bar{x}| - G(\bar{x},\bar{\lambda}))(\xi_1) \times \{0^m\} \\ &\quad + \begin{pmatrix} J_xG(x,\lambda)^\top & -J\tilde{g}_\beta(\bar{x})^\top & 0 \\ J_\lambda G(\bar{x},\bar{\lambda})^\top & 0 & I_{C(A)}^\top \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \\ &= (D^*F(\bar{x}| - G(\bar{x},\bar{\lambda}))(\xi_1) + J_xG(\bar{x},\bar{\lambda})^\top\xi_1 - J\tilde{g}_\beta(\bar{x})^\top\xi_2) \\ &\quad \times \{J_\lambda G(\bar{x},\bar{\lambda})^\top\xi_1 + I_{C(A)}^\top\xi_3\}. \end{aligned}$$

with $I_{C(A)} \in \mathbb{R}^{m\times|C(A)|}$ the matrix satisfying, for all $y \in \mathbb{R}^m$, $I_{C(A)}y = y_{C(A)}$. This matrix satisfies the following properties:

$$\forall y \in \mathbb{R}^{|C(A)|},\ (I_{C(A)}^\top y)_A = 0 \text{ and } (I_{C(A)}^\top y)_{C(A)} = y. \tag{21}$$

Suppose that $0 \in D^*\Phi_\beta(\bar{z}|0)(\xi)$. This implies that

$$0 \in D^*F(\bar{x}| - G(\bar{x},\bar{\lambda}))(\xi_1) + J_xG(\bar{x},\bar{\lambda})^\top\xi_1 - J\tilde{g}_\beta(\bar{x})^\top\xi_2 \tag{22}$$

and

$$J_\lambda G(\bar{x},\bar{\lambda})^\top\xi_1 + I_{C(A)}^\top\xi_3 = 0. \tag{23}$$

Since $(I_{C(A)}^\top \xi)_A = 0$, we have $(I_{C(A)}^\top \xi)_\beta = 0$ because $\beta \subset A$. Therefore, we have $(J_\lambda G(\bar{x},\bar{\lambda})^\top \xi_1)_\beta = 0$.

An easy calculus permits us to obtain that

$$(J_\lambda G(\bar{x},\bar{\lambda})^\top \xi_1)_\beta = \begin{pmatrix} J_{x^1} g^1_{\beta^1}(\bar{x})\xi_1^1 \\ \vdots \\ J_{x^p} g^p_{\beta^p}(\bar{x})\xi_1^p \end{pmatrix}.$$

Finally, the condition $(J_\lambda G(\bar{x},\bar{\lambda})\xi_1)_\beta^\top = 0$ is reduced to

$$\forall \nu \in \{1,\dots,p\}, \ J_{x^\nu} g^\nu_{\beta^\nu}(\bar{x})\xi_1^\nu = 0. \tag{24}$$

Therefore, from (22) and (24), assumption (25) implies that $\xi_1 = 0$ and $\xi_2 = 0$.

Since $\xi_1 = 0$, we have $I_{C(A)}^\top \xi_3 = 0$ by (23); then, $\xi_3 = (I_{C(A)}^\top \xi_3)_{C(A)} = 0$ by (21). Finally, $\xi = 0$. Therefore, by [9, Corollary 3.8], Φ_β is metrically regular near $(\bar{z}, 0)$. □

We are now in a position to give a result for the metric regularity of Φ.

Theorem 4.3: *Suppose that the strict complementarity assumption holds for a given $\bar{z} \in \Omega$, and that*

$$\forall \xi := (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^{|A|},$$

$$\left.\begin{array}{l} 0 \in D^*F(\bar{x}| - G(\bar{x},\bar{\lambda}))(\xi_1) + J_x G(\bar{x},\bar{\lambda})^\top \xi_1 - J\tilde{g}_A(\bar{x})^\top \xi_2 \\ \forall \nu \in \{1,\dots,p\}, \ J_{x^\nu} g^\nu_{A^\nu}(\bar{x})\xi_1^\nu = 0 \ \text{if}\ A^\nu \neq \emptyset \end{array}\right\} \Rightarrow \xi = 0, \tag{25}$$

with $\xi_1 = (\xi_1^\nu)_{\nu=1,\dots,p}$ and $\xi_1^\nu \in \mathbb{R}^{n_\nu}$.

Then, the set-valued mapping Φ is metrically regular near $(\bar{z}, 0)$.

Proof: We can observe that $A \cup C(A) = \{1,\dots,m\}$ and $\bar{\lambda}_A > 0$ since the assumption of strict complementarity holds. Therefore, the above theorem is a direct consequence of Lemma 4.2 with $\beta = A$. □

Since metric regularity implies metric subregularity, the assumptions of Theorem 4.3 imply that Φ is metrically subregular at $(\bar{z}, 0)$, which implies that the following error bound property holds at the point $\bar{z}$:

$$\exists r, L > 0, \ \forall z \in B(\bar{z}, r), \ \operatorname{dist}(z, \Omega) \leq L\operatorname{dist}(0, \Phi(z)).$$

In the case where the functions $\{\theta^\nu : \nu = 1,\dots,p\}$ are C^2 around $\bar{x}$, we can observe that the assumption (25) is equivalent to the non-singularity of the matrix

$$\begin{pmatrix} J_x L(\bar{x},\lambda) & E_A(\bar{x}) \\ J\tilde{g}_A(\bar{x}) & 0 \end{pmatrix}$$

with $E_A(\bar{x}) := \operatorname{diagblock}\{\nabla_{x^\nu} g^\nu_{A^\nu}(\bar{x}) : \nu = 1,\dots,p\}$, that is a standard assumption in the study of stability of the GNEP (see e.g. [2–4]).

When two or more players share the same active constraint, then it is impossible that assumption (25) holds because $J\tilde{g}_A(\bar{x})^\top$ has two or more equal columns. This situation occurs when the players have to share the same resource.

Consider $\bar{z} = (\bar{x}, \bar{\lambda}) \in \Omega$. As in [2], we can consider a family of functions $(\tilde{g}_i)_{i\in\alpha}$ with $\alpha \subset A$ satisfying the following property: for all $(i,j) \in \alpha \times \alpha$, if $i \neq j$, then $\tilde{g}_i \neq \tilde{g}_j$ in any neighbourhood of $\bar{x}$, and for any $i \in A \setminus \alpha$, there exists $j \in \alpha$ such that $\tilde{g}_i = \tilde{g}_j$ in a neighbourhood of $\bar{x}$.

What follows permits us to formally define sets of indices α satisfying the above property. We first define $\mathcal{J}(\bar{x})$ as the sets of indices such that two active constraint functions cannot be equal in any neighbourhood of $\bar{x}$.

$$\mathcal{J}(\bar{x}) := \{\beta \subset A \ : \ \forall (i,j) \in \beta \times \beta \ , \ i \neq j \ , \ \tilde{g}_i \neq \tilde{g}_j \text{ in any neighbourhood of } \bar{x}\}.$$

The sets of indices that we wanted to define formally are the elements of $\mathcal{J}(\bar{z})$ maximizing the cardinality; more precisely,

$$\mathcal{Q}(\bar{x}) := \{\alpha \in \mathcal{J}(\bar{x}) \ : \ |\alpha| = \max_{\beta \in \mathcal{J}(\bar{x})} |\beta|\}. \tag{26}$$

We can observe that for all $\alpha \in \mathcal{Q}(\bar{x})$, if $\alpha \neq A$, then there exists $r > 0$ such that for all $i \in A \setminus \alpha$, there exists $j \in \alpha$ satisfying $\tilde{g}_i = \tilde{g}_j$ on $B(\bar{x}, r)$.

Theorem 4.4: *Let $\bar{z} \in \Omega$. Suppose there exists a set of indices $\alpha \in \mathcal{Q}(\bar{x})$, with $\mathcal{Q}(\bar{x})$ defined in (26), such that $\bar{\lambda}_\alpha > 0$ and the following condition holds:*

$$\forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^{|\alpha|},$$

$$\left.\begin{array}{l} 0 \in D^*F(\bar{x}| - G(\bar{x},\bar{\lambda}))(\xi_1) + J_x G(\bar{x},\bar{\lambda})^\top \xi_1 - J\tilde{g}_\alpha(\bar{x})^\top \xi_2 \\ \forall \nu \in \{1,\dots,p\},\ J_{x^\nu} g^\nu_{\alpha^\nu}(\bar{x})\xi_1^\nu = 0 \ \textit{if}\ \alpha^\nu \neq \emptyset \end{array}\right\} \Rightarrow \xi = 0, \tag{27}$$

with $\xi_1 = (\xi_1^\nu)_{\nu=1,\dots,p}$, $\xi_1^\nu \in \mathbb{R}^{n_\nu}$ and α^ν defined by (12).

Then, the set-valued mapping Φ is metrically subregular at $(\bar{z}, 0)$.

Proof: Define $\Phi_\alpha : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+|\alpha|+|C(A)|}$ by

$$\Phi_\alpha(x,\lambda) := \begin{pmatrix} T(x,\lambda) \\ -\tilde{g}_\alpha(x) \\ \lambda_{C(A)} \end{pmatrix}$$

Consider the set of indices $\mathcal{I}' := \alpha \cup C(A)$. We can observe that, after some permutations of the coordinates of $(\tilde{g}_i)_{i\in\mathcal{I}}$, for all $z \in \mathbb{R}^{n+m}$ in some neighbourhood of $\bar{z}$,

$$\Phi_\alpha(z) = \begin{pmatrix} T(z) \\ (\min\{\lambda_i, -\tilde{g}_i(x)\})_{i\in\mathcal{I}'} \end{pmatrix}$$

because the function $\tilde{g}_i$ is continuous at $\bar{x}$ for any $i \in \mathcal{I}$ and $\bar{\lambda}_i > 0$ for any $i \in \alpha$. The strict complementarity assumption holds for the set of indices $\mathcal{I}'$ at the point $\bar{z}$ (i.e. for all $i \in \mathcal{I}'$, $\tilde{g}_i(\bar{x}) < 0$ or $\bar{\lambda}_i > 0$); then, the assumption (27) implies, by Lemma 4.2, that Φ_α is metrically regular near $(\bar{z}, 0)$, so Φ_α is metrically subregular at $(\bar{z}, 0)$. Therefore, we can choose $r_1 > 0$ small enough such that for all $z \in B(\bar{z}, r_1)$,

$$\operatorname{dist}(z, \Omega_\alpha) \leq L\operatorname{dist}(0, \Phi_\alpha(z))$$

where $\Omega_\alpha := \Phi_\alpha^{-1}(0)$. We can easily verify that $\operatorname{dist}(0, \Phi_\alpha(z)) \leq \operatorname{dist}(0, \Phi(z))$; thus, for all $z \in B(\bar{z}, r_1)$, the following inequality holds:

$$\operatorname{dist}(z, \Omega_\alpha) \leq L\operatorname{dist}(0, \Phi(z)). \tag{28}$$

We now prove that there exists a real $r_2 > 0$ such that for all $z \in B(\bar{z}, r_2)$, $\operatorname{dist}(z, \Omega_\alpha) = \operatorname{dist}(z, \Omega)$. We first show that there exists a real $r > 0$ such that $\Omega \cap B(\bar{z}, r) = \Omega_\alpha \cap B(\bar{z}, r)$.

Let $r > 0$ be such that for all $(x, \lambda) \in B(\bar{z}, r)$, $\lambda_\alpha > 0$ and $\tilde{g}_{C(A)}(x) < 0$ (r exists since by assumption, $\bar{\lambda}_\alpha > 0$ and $\tilde{g}$ is continuous at $\bar{x}$). For all $z \in \Omega \cap B(\bar{z}, r)$, we have $\tilde{g}_\alpha(x) = 0$ since $\lambda_\alpha > 0$, and $\lambda_{C(A)} = 0$ since $\tilde{g}_{C(A)}(x) < 0$. Therefore, $\Omega \cap B(\bar{z}, r) \subset \Omega_\alpha \cap B(\bar{z}, r)$. We must still show that $\Omega_\alpha \cap B(\bar{z}, r) \subset \Omega \cap B(\bar{z}, r)$ for r small enough.

Suppose r is small enough such that for all $i \in A \setminus \alpha$, there exists $j \in \alpha$ such that $\tilde{g}_i(x) = \tilde{g}_j(x)$, for all $(x, \lambda) \in B(\bar{z}, r)$ (we can choose such a real r because $\alpha \in \mathcal{Q}(\bar{x})$).

Let $z \in \Omega_\alpha \cap B(\bar{z}, r)$. We know that $0 \in T(z)$, $\tilde{g}_\alpha(x) = 0$ and $\lambda_{C(A)} = 0$; thus, we have to prove that $\tilde{g}_{A\setminus\alpha}(x) = 0$. For all $i \in A \setminus \alpha$, there exists $j \in \alpha$ such that $\tilde{g}_i(x) = \tilde{g}_j(x) = 0$, which shows that $z \in \Omega \cap B(\bar{z}, r)$.

Take $r_2 < r/2$ and $z \in B(\bar{z}, r_2)$. Since $\bar{z} \in \Omega_\alpha$, $\mathrm{dist}(z, \Omega_\alpha) \leq r_2$. Let $w \in \Omega_\alpha$ be such that $\|z - w\| = \mathrm{dist}(z, \Omega_\alpha)$. We have $\|z - w\| \leq r_2$; thus, $\|w - \bar{z}\| \leq \|w - z\| + \|z - \bar{z}\| \leq 2r_2 < r$ and then $w \in \Omega_\alpha \cap B(\bar{z}, r)$. Therefore, we obtain

$$\mathrm{dist}(z, \Omega_\alpha) = \min_{w \in \Omega_\alpha \cap B(\bar{z}, r)} \|z - w\|, \quad \forall z \in B(\bar{z}, r_2).$$

In the same way, we have

$$\mathrm{dist}(z, \Omega) = \min_{w \in \Omega \cap B(\bar{z}, r)} \|z - w\|, \quad \forall z \in B(\bar{z}, r_2).$$

Since $\Omega \cap B(\bar{z}, r) = \Omega_\alpha \cap B(\bar{z}, r)$, for all $z \in B(\bar{z}, r_2)$, $\mathrm{dist}(z, \Omega_\alpha) = \mathrm{dist}(z, \Omega)$. Finally, from (28) we obtain that

$$\forall z \in B(\bar{z}, \min\{r_1, r_2\}) \text{ , } \mathrm{dist}(z, \Omega) \leq L\mathrm{dist}(0, \Phi(z)),$$

which proves that Φ is metrically subregular at $(\bar{z}, 0)$ since $\Omega = \Phi^{-1}(0)$. □

We can observe that the previous theorem does not imply the metric regularity of Φ at the reference point, as shows the following example.

Example 4.5: Consider the following Nash game:

Player 1:

$$\min_{x_1} x_1 \text{ s. t. } x_1 + x_2 \geq 0$$

and

Player 2:

$$\min_{x_2} 0.5|x_2| \text{ s. t. } x_1 + x_2 \geq 0$$

The set-valued mapping Φ is:

$$\Phi(x, \lambda) = \begin{pmatrix} 1 - \lambda_1 \\ 0.5\partial|\cdot|(x_2) - \lambda_2 \\ \min\{-x_1 - x_2, \lambda_1\} \\ \min\{-x_1 - x_2, \lambda_2\} \end{pmatrix}$$

We have $0 \in \Phi(\bar{z})$ with $\bar{z} = (0, 0, 1, 0)$. We first verify that the hypotheses of Theorem 4.4 are satisfied at $\bar{z}$. Since $\bar{\lambda}_1 > 0$, we choose $\alpha = \{1\}$. Since $\alpha^1 = \{1\}$ and $\alpha^2 = \emptyset$, where α^ν has been defined in (12), the assumptions of Theorem 4.4 are reduced

$$\forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \times \mathbb{R},$$

$$\left.\begin{array}{c} 0 \in D^*F(0, 0|1, 0)(\xi_1) + \begin{pmatrix} \xi_2 \\ \xi_2 \end{pmatrix} \\ \xi_1^1 = 0 \end{array}\right\} \Rightarrow \xi = 0,$$

where $F(x_1, x_2) = \{1\} \times \partial(|\cdot|)(x_2)$. We have:

$$D^*F(0, 0|1, 0)(\xi_1) = \begin{cases} \{0\} \times \mathbb{R} & \text{if } \xi_1^2 = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

Therefore, $\xi_2 = 0$ and $\xi_1^2 = 0$, which implies by Theorem 4.4 that Φ is metrically subregular at $(0, 0, 1, 0)$.

We now prove that Φ is not metrically regular around $(0,0,1,0)$. Consider $y=(0,0,0,\varepsilon)$, with $\varepsilon>0$, and $z=(x,\lambda)\in\Phi^{-1}(y)$. We must have $\lambda_1=1$. Since $\min\{-x_1-x_2,\lambda_1\}=0$ and $\lambda_1>0$, we have $x_1+x_2=0$. But in this case $\varepsilon=\min\{-x_1-x_2,\lambda_2\}\leq 0$ which contradicts $\varepsilon>0$, then $\Phi^{-1}(y)=\emptyset$, which implies that $\operatorname{dist}(z,\Phi^{-1}(y))=+\infty$, for all $z\in\mathbb{R}^4$, which implies that the inequality

$$\operatorname{dist}(z,\Phi^{-1}(y))\leq L\operatorname{dist}(y,\Phi(z))$$

cannot occur with any $L>0$ and $z\in\mathbb{R}^4$; therefore, Φ is not metrically regular around $(0,0,1,0)$.

5. Application to stability in the GNEP

In this section, we suppose that the functions θ^ν and g^ν depend on a parameter u. More precisely, let us consider a metric space (U,δ), and suppose that $\theta^\nu:\mathbb{R}^n\times U\to\mathbb{R}$ and $g^\nu:\mathbb{R}^n\times U\to\mathbb{R}^{m^\nu}$. Moreover, we make the assumption that θ^ν is locally Lipschitz on $\mathbb{R}^n\times U$, $g^\nu(\cdot,u)$ is C^2 on $\mathbb{R}^n$ and for all $u\in U$ g^ν and $\nabla_{x^\nu}g^\nu$ are continuous on $\mathbb{R}^n\times U$.

Define the set-valued mapping $S:U\rightrightarrows\mathbb{R}^n$ by:

$$S(u):=\left\{\overline{x}\in\mathbb{R}^n\ \middle|\ \begin{array}{l}\forall\nu\in\{1,\dots,p\},\ \overline{x}^\nu\in X_\nu(\overline{x}^{-\nu},u)\text{ and}\\ \theta^\nu(\overline{x}^\nu,\overline{x}^{-\nu},u)=\displaystyle\min_{x^\nu\in X_\nu(\overline{x}^{-\nu},u)}\theta^\nu(x^\nu,\overline{x}^{-\nu},u)\end{array}\right\}\tag{29}$$

with, for all $\nu\in\{1,\dots,p\}$,

$$X_\nu(x^{-\nu},u):=\{x^\nu\in\mathbb{R}^{n^\nu}\ :\ g^\nu(x^\nu,x^{-\nu},u)\leq 0\}.\tag{30}$$

For every player $\nu\in\{1,\dots,p\}$, we introduce the Lagrangian function $L^\nu:\mathbb{R}^{n^\nu}\times\mathbb{R}^{n^{-\nu}}\times\mathbb{R}^{m^\nu}\times U\to\mathbb{R}$ which is given by

$$L^\nu(x^\nu,x^{-\nu},\lambda^\nu,u):=\theta^\nu(x^\nu,x^{-\nu},u)+{}^t\lambda^\nu g^\nu(x^\nu,x^{-\nu},u).$$

Therefore, $\bar{x}=(\bar{x}^1,\dots,\bar{x}^p)\in S(u)$ if and only if there exists a vector $\bar{\lambda}=(\bar{\lambda}^1,\dots,\bar{\lambda}^p)\in\mathbb{R}^m$ satisfying, for all $\nu\in\{1,\dots,p\}$,

$$0\in\partial_{x^\nu}L(\bar{x}^\nu,\bar{x}^{-\nu},\bar{\lambda}^\nu,u)\text{ and }0\leq\bar{\lambda}^\nu\perp(-g^\nu(\bar{x}^\nu,\bar{x}^{-\nu},u))\geq 0\tag{31}$$

We introduce the set-valued mapping $\Phi:\mathbb{R}^n\times\mathbb{R}^m\times U\rightrightarrows\mathbb{R}^n\times\mathbb{R}^m$ defined by

$$\forall z=(x,\lambda)\in\mathbb{R}^n\times\mathbb{R}^m,\ \Phi(z,u):=\begin{pmatrix}T(z,u)\\ \min\{\lambda^1,-g^1(x^1,x^{-1},u)\}\\ \vdots\\ \min\{\lambda^p,-g^p(x^p,x^{-p},u)\}\end{pmatrix}\tag{32}$$

with

$$T(z,u):=\partial_{x^1}L^1(x^1,x^{-1},\lambda^1,u)\times\cdots\times\partial_{x^p}L^p(x^p,x^{-p},\lambda^p,u).\tag{33}$$

A vector $\bar{x}=(\bar{x}^1,\dots,\bar{x}^p)\in S(u)$ if and only if there exists a vector $\bar{\lambda}=(\bar{\lambda}^1,\dots,\bar{\lambda}^p)\in\mathbb{R}^m$ satisfying

$$0\in\Phi(\bar{z},u)\text{ with }\bar{z}=(\bar{x},\bar{\lambda}).\tag{34}$$

We finally introduce the solution map $\Omega:U\rightrightarrows\mathbb{R}^{n+m}$ by

$$\Omega(u):=\left\{z\in\mathbb{R}^{n+m}\ :\ 0\in\Phi(z,u)\right\}.\tag{35}$$

We introduce the set of Lagrange multipliers at any point $(u, x) \in \mathrm{Gr}(S)$:

$$\forall (u, x) \in \mathrm{Gr}(S),\ \Lambda(x, u) := \{\lambda \in \mathbb{R}^m\ :\ (x, \lambda) \in \Omega(u)\}. \tag{36}$$

The following lemma shows that Ω has a closed graph.

Lemma 5.1: *The set-valued mapping Ω has a closed graph.*

Proof: Let a sequence $(u_n, z_n)_n \in (\mathrm{Gr}(\Omega))^{\mathbb{N}}$ converge to $(\bar{u}, \bar{z})$. For all $n \in \mathbb{N}$, $0 \in \Phi(z_n, u_n)$. In the same way, as in the proof of Lemma 3.3, using the continuity of the functions θ^ν, g^ν and $\nabla_{x^\nu} g^\nu$, we can prove that Φ has a closed graph. Since $(z_n, u_n, 0)_n \in (\mathrm{Gr}(\Phi))^{\mathbb{N}}$, we deduce that $(\bar{z}, \bar{u}, 0) \in \mathrm{Gr}(\Phi)$, thus $(\bar{u}, \bar{z}) \in \mathrm{Gr}(\Omega)$, proving that Ω has a closed graph. □

The following proposition relates an error bound property around $\bar{z} \in \Omega(\bar{u})$ with the calmness of Ω at $(\bar{u}, \bar{x})$:

Proposition 5.2: *Let $\bar{u} \in U$ and $\bar{z} \in \Omega(\bar{u})$. Suppose that there exist positive reals $r, r_1, r_2, \varepsilon, L, L'$ such that the following assumptions hold:*

(1) *For all $z \in B(\bar{z}, r)$,*

$$\mathit{dist}(z, \Omega(\bar{u})) \leq L\mathit{dist}(0, \Phi(z, \bar{u})).$$

(2) *For all $z \in B(\bar{z}, r_1)$, for all $u \in B(\bar{u}, r_2)$,*

$$\Phi(z, u) \cap B(0, \varepsilon) \subset \Phi(z, \bar{u}) + B(0, L'\delta(u, \bar{u})). \tag{37}$$

Then, for all $u \in B(\bar{u}, r_2)$,

$$\Omega(u) \cap B(\bar{z}, \min\{r, r_1\}) \subset \Omega(\bar{u}) + B(0, LL'\delta(u, \bar{u})).$$

Proof: Let $u \in B(\bar{u}, r_2)$, and $z \in \Omega(u) \cap B(\bar{z}, \min\{r, r_1\})$. One has $0 \in \Phi(z, u)$; then, by assumption (37), one has $0 \in \Phi(z, \bar{u}) + B(0, L'\delta(u, \bar{u}))$. There exists $d \in B(0, L'\delta(u, \bar{u}))$ such that $d \in \Phi(z, \bar{u})$. Finally, $\mathrm{dist}(z, \Omega(\bar{u})) \leq L\mathrm{dist}(0, \Phi(z, \bar{u})) \leq L\|d\| \leq LL'\delta(u, \bar{u})$. That concludes the proof. □

The following proposition gives a family of pay-off functions θ^ν such that assumption (37) holds.

Proposition 5.3: *Let $\bar{u} \in U$ and $\bar{z} \in \Omega(\bar{u})$. Suppose that there exists a constant $r > 0$ such that for all players ν, there exist two functions $\theta^{\nu,1} : B(\bar{x}, r) \to \mathbb{R}$ and $\theta^{\nu,2} : B(\bar{x}, r) \times B(\bar{u}, r) \to \mathbb{R}$ such that $\theta^{\nu,2}(\cdot, u)$ is differentiable on $B(\bar{x}, r)$ for all $u \in B(\bar{u}, r)$ and for all $x \in B(\bar{x}, r)$, for all $u \in B(\bar{u}, r)$, one has*

$$\theta^\nu(x, u) = \theta^{\nu,1}(x) + \theta^{\nu,2}(x, u).$$

Assume $\nabla_{x^\nu}\theta^{\nu,2}$, g^ν and $\nabla_{x^\nu} g^\nu$ are locally Lipschitz on $B(\bar{x}, r) \times B(\bar{u}, r)$. Then, assumption (37) holds.

Proof: For a better understanding of this proof, we recall that $T(z, u)$ has been defined in (33).

For all $(x, \lambda, u) \in B(\bar{x}, r) \times \mathbb{R}^m \times B(\bar{u}, r)$, for all players ν, we have

$$L^\nu(x, \lambda^\nu, u) = \theta^{\nu,1}(x) + \theta^{\nu,2}(x, u) +^t \lambda^\nu g^\nu(x, u) = L^{\nu,1}(x) + L^{\nu,2}(x, \lambda, u)$$

where $L^{\nu,1}(x) := \theta^{\nu,1}(x)$ and $L^{\nu,2}(x, \lambda, u) := \theta^{\nu,2}(x, u) +^t \lambda^\nu g^\nu(x, u)$. By assumption, $L^{\nu,2}$ is C^2 on $B(\bar{x}, r) \times \mathbb{R}^m \times B(\bar{u}, r)$; thus, we have, for all $(z, u) = (x, \lambda, u) \in B(\bar{x}, r) \times \mathbb{R}^m \times B(\bar{u}, r)$:

$$\begin{aligned} T(z, u) &= (\partial_{x^1} L^{1,1}(x^1, x^{-1}) + \nabla_{x^1} L^{1,2}(x^1, x^{-1}, \lambda^1, u)) \times \cdots \times \\ &\quad (\partial_{x^p} L^{p,1}(x^p, x^{-p}) + \nabla_{x^p} L^{p,2}(x^p, x^{-p}, \lambda^p, u)) \\ &= T^1(x) + T^2(x, \lambda, u) \end{aligned}$$

with

$$T^1(x) := \partial_{x^1} L^{1,1}(x^1, x^{-1}) \times \cdots \times \partial_{x^p} L^{p,1}(x^p, x^{-p})$$

and

$$T^2(x, \lambda, u) := \begin{pmatrix} \nabla_{x^1} L^{1,2}(x^1, x^{-1}, \lambda^1, u) \\ \vdots \\ \nabla_{x^p} L^{p,2}(x^p, x^{-p}, \lambda^p, u \end{pmatrix}$$

Let the set-valued mapping $\Phi^1 : B(\bar{x}, r) \times \mathbb{R}^m \rightrightarrows \mathbb{R}^{n+m}$ and the single-valued map $\Phi^2 : B(\bar{x}, r) \times \mathbb{R}^m \times B(\bar{u}, r) \to \mathbb{R}^{n+m}$ be defined, for all $(z, u) = (x, \lambda, u) \in B(\bar{x}, r) \times \mathbb{R}^m \times B(\bar{u}, r)$, by:

$$\Phi^1(z) := T^1(x) \times \{0^m\} \qquad \text{and} \qquad \Phi^2(z, u) := \begin{pmatrix} T^2(z, u) \\ \min\{\lambda^1, -g^1(x^1, x^{-1}, u)\} \\ \vdots \\ \min\{\lambda^p, -g^p(x^p, x^{-p}, u)\} \end{pmatrix}$$

We observe that

$$\Phi(z, u) = \Phi^1(z) + \Phi^2(z, u).$$

Since $\nabla_{x^\nu} \theta^{\nu,2}$, g^ν and $\nabla_{x^\nu} g^\nu$ are locally Lipschitz on $B(\bar{x}, r) \times B(\bar{u}, r)$, we deduce that Φ^2 is locally Lipschitz around $(\bar{z}, \bar{u})$.

Therefore, there exist $r', L > 0$ such that $r' < r$ and for all $z \in B(\bar{z}, r')$, for all $u \in B(\bar{u}, r')$, $\Phi^2(z, u) \in \Phi^2(z, \bar{u}) + B(0, L\delta(u, \bar{u}))$. Adding $\Phi^1(z)$, we finally obtain that

$$\Phi(z, u) \subset \Phi(z, \bar{u}) + B(0, L\delta(u, \bar{u})).$$

□

We can directly deduce a result of calmness for the KKT system associated with the GNEP.

Proposition 5.4: *Let $\bar{u} \in U$ and $\bar{z} \in \Omega(\bar{u})$. Suppose that the assumptions of Theorem 4.3 or the assumptions of Theorem 4.4 are satisfied at $\bar{z}$ for the unperturbated game $\mathcal{G}(\bar{u}) = ((\theta^\nu(\cdot, \bar{u}))_\nu, (g^\nu(\cdot, \bar{u}))_\nu)$. Suppose moreover that assumption (37) is satisfied at $(\bar{u}, \bar{z})$. Then, there exist constants $\varepsilon, r, L > 0$ such that, for all $u \in B(\bar{u}, \varepsilon)$, one has:*

$$\Omega(u) \cap B(\bar{z}, r) \subset \Omega(\bar{u}) + B(0, L\delta(u, \bar{u})). \tag{38}$$

Proof: From Theorems 4.3 and 4.4, Φ is metrically subregular at $(\bar{z}, 0)$; then, Assumption 1 in Proposition 5.2 holds. Since Assumption (37) is satisfied at $(\bar{u}, \bar{z})$, we deduce from Proposition 5.2 that Ω is calm at $(\bar{u}, \bar{z})$. □

A natural question is to ask if the result of calmness of the KKT system implies the calmness of the primal problem. The next proposition gives an answer to this question under the assumption that Slater condition is satisfied.

Proposition 5.5: *Let $\bar{u} \in U$ and $\bar{x} \in S(\bar{u})$ be such that for any $\bar{\lambda} \in \Lambda(\bar{x}, \bar{u})$, there exist constants $r, \varepsilon > 0$, $L \geq 0$ such that, for all $u \in B(\bar{u}, \varepsilon)$,*

$$\Omega(u) \cap B(\bar{z}, r) \subset \Omega(\bar{u}) + B(0, L\delta(u, \bar{u})),$$

with $\bar{z} = (\bar{x}, \bar{\lambda})$. Moreover, we suppose that the functions θ^ν are locally Lipschitz on $\mathbb{R}^n \times U$ and for every player $\nu \in \{1, \ldots, p\}$, there exists an element y^ν such that

$$g^\nu(y^\nu, \bar{x}^{-\nu}, \bar{u}) < 0.$$

Then, there exist constants $r', \varepsilon' > 0$, $L' \geq 0$ *such that, for all* $u \in B(\bar{u}, \varepsilon')$,

$$S(u) \cap B(\bar{x}, r') \subset S(\bar{u}) + B(0, L'\delta(u, \bar{u})).$$

Proof: We first observe that if S is not calm at the point $(\bar{u}, \bar{x})$, then there exists a sequence $(u_n, x_n)_n \in \mathrm{Gr}(S)^{\mathbb{N}}$ such that $(u_n, x_n) \to (\bar{u}, \bar{x})$, $u_n \neq \bar{u}$ for any integer n and $\lim_{n\to+\infty} \dfrac{\mathrm{dist}(x_n, S(\bar{u}))}{\delta(u_n, \bar{u})} = +\infty$. Therefore, if we show that for any sequence $(u_n, x_n)_n \in \mathrm{Gr}(S)^{\mathbb{N}}$ such that $(u_n, x_n) \to (\bar{u}, \bar{x})$ and $u_n \neq \bar{u}$ for any integer n, there exists a subsequence $(u_{n_k}, x_{n_k})_k$ such that the sequence $\left(\dfrac{\mathrm{dist}(x_{n_k}, S(\bar{u}))}{\delta(u_{n_k}, \bar{u})}\right)_k$ is bounded; then, S is calm at $(\bar{u}, \bar{x})$. That is the idea of the proof.

Let $(u_n, x_n)_n \in \mathrm{Gr}(S)^{\mathbb{N}}$ be such that $(u_n, x_n) \to (\bar{u}, \bar{x})$ and $u_n \neq \bar{u}$ for any integer n. For every $n \in \mathbb{N}$, let $\lambda_n \in \Lambda(x_n, u_n)$. We show that the sequence $(\lambda_n)_n$ is bounded. Suppose that $(\lambda_n)_n$ is not a bounded sequence. Then, there exists a subsequence $(\lambda_{n_k})_k$ and a player $\nu \in \{1, \ldots, p\}$ such that $\lim_{k\to+\infty} \|\lambda^{\nu}_{n_k}\| = +\infty$.

Since $(x_{n_k}, \lambda_{n_k}) \in \Omega(u_{n_k})$, one has

$$0 \in \partial_{x^\nu}\theta^\nu(x^\nu_{n_k}, x^{-\nu}_{n_k}, u_{n_k}) + J_{x^\nu} g^\nu(x^\nu_{n_k}, x^{-\nu}_{n_k}, u_{n_k})^\top \lambda^\nu_{n_k},$$

and there exists an element $v^\nu_{n_k} \in \partial_{x^\nu}\theta^\nu(x^\nu_{n_k}, x^{-\nu}_{n_k}, u_{n_k})$ such that

$$v^\nu_{n_k} + J_{x^\nu} g^\nu(x^\nu_{n_k}, x^{-\nu}_{n_k}, u_{n_k})^\top \lambda^\nu_{n_k} = 0.$$

Since the function θ^ν is locally Lipschitz by assumption, the sequence $(v^\nu_{n_k})_k$ is bounded; thus, $\lim_{k\to+\infty} \dfrac{v^\nu_{n_k}}{\|\lambda^\nu_{n_k}\|} = 0$. Dividing by $\|\lambda^\nu_{n_k}\|$ in the above equality, and taking a subsequence such that the sequence $\left(\dfrac{\lambda^\nu_{n_{k_l}}}{\|\lambda^\nu_{n_{k_l}}\|}\right)_l$ converges to an element $\bar{\lambda}^\nu$, we obtain

$$J_{x^\nu} g^\nu(\bar{x}^\nu, \bar{x}^{-\nu}, \bar{u})^\top \bar{\lambda}^\nu = 0. \tag{39}$$

Let $i \notin A^\nu(\bar{x})$ since $g^\nu_i(\bar{x}, \bar{u}) < 0$ and g^ν_i is continuous at $\bar{x}$; for all l large enough, one has $g^\nu_i(x_{n_{k_l}}, u_{n_{k_l}}) < 0$; thus, $\lambda^\nu_{i,n_{k_l}} = 0$, and by passing at the limit, $\bar{\lambda}^\nu_i = 0$. Therefore, equality (39) becomes

$$\sum_{i\in A^\nu(\bar{x})} \bar{\lambda}^\nu_i \nabla g^\nu_i(\bar{x}, \bar{u}) = 0,$$

with $\bar{\lambda}^\nu_i \geq 0$, for all $i \in A^\nu(\bar{x})$, since $\lambda^\nu_{i,n_{k_l}} \geq 0$ for all $i \in A^\nu(\bar{x})$ and $l \in \mathbb{N}$.

By assumption, the Slater constraint qualification holds for player ν; thus, the Mangasarian–Fromovitz constraint qualification holds at $\bar{x}^\nu$ for player ν because $g^\nu(\cdot, \bar{x}^{-\nu}, \bar{u})$ is a convex function (see e.g. [18]). Therefore, the above equality implies that $\bar{\lambda}^\nu_i = 0$, for all $i \in A^\nu(\bar{x})$, which implies that $\bar{\lambda}^\nu = 0$. That is a contradiction with the equality $\|\bar{\lambda}^\nu\| = 1$. We conclude that the sequence $(\lambda_n)_n$ is bounded.

Take a subsequence $(\lambda_{n_k})_k$ converging to an element $\bar{\lambda}$. Since by Lemma 5.1 Ω has a closed graph and $(u_{n_k}, x_{n_k}, \lambda_{n_k}) \in \mathrm{Gr}(\Omega)$, we deduce that $(\bar{x}, \bar{\lambda}) \in \Omega(\bar{u})$. By assumption, there exist constants $r, \varepsilon > 0$, $L \geq 0$ such that, for all $u \in B(\bar{u}, \varepsilon)$,

$$\Omega(u) \cap B((\bar{x}, \bar{\lambda}), r) \subset \Omega(\bar{u}) + B(0, L\delta(u, \bar{u})).$$

Since $u_{n_k} \in B(\bar{u}, \varepsilon)$ and $(x_{n_k}, \lambda_{n_k}) \in B((\bar{x}, \bar{\lambda}), r)$ for all k large enough, we have

$$\frac{\operatorname{dist}((x_{n_k}, \lambda_{n_k}), \Omega(\bar{u}))}{\delta(u_{n_k}, \bar{u})} \leq L.$$

Let $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ and $(p_1, p_2) \in \Omega(\bar{u})$ be such that

$$\operatorname{dist}((x, \lambda), \Omega(\bar{u})) = \|(x, \lambda) - (p_1, p_2)\|.$$

Since $p_1 \in S(\bar{u})$, we have:

$$\begin{aligned}
\operatorname{dist}(x, S(\bar{u})) &\leq \|x - p_1\| \\
&= \sqrt{\sum_{k=1}^{n} (x_k - p_{1,k})^2} \\
&\leq \sqrt{\sum_{k=1}^{n} (x_k - p_{1,k})^2 + \sum_{k=1}^{m} (\lambda_k - p_{2,k})^2} \\
&= \|(x, \lambda) - (p_1, p_2)\| \\
&= \operatorname{dist}((x, \lambda), \Omega(\bar{u})).
\end{aligned}$$

Therefore, for all k large enough, we have

$$\frac{\operatorname{dist}(x_{n_k}, S(\bar{u}))}{\delta(u_{n_k}, \bar{u})} \leq L,$$

which proves that the sequence $\left(\frac{\operatorname{dist}(x_{n_k}, S(\bar{u}))}{\delta(u_{n_k}, \bar{u})}\right)_k$ is bounded, and implies the calmness of S at $(\bar{u}, \bar{x})$. □

6. A generic example with a family of two-player games

In this section, we consider a two-player game. We suppose that the players 1 and 2 manage loss functions $\theta^1 : \mathbb{R}^2 \to \mathbb{R}$ and $\theta^2 : \mathbb{R}^2 \to \mathbb{R}$ which have the following form:

$$\theta^1(x^1, x^2) = \theta^{1,1}(x^1) + \theta^{1,2}(x^1, x^2), \qquad \theta^2(x^1, x^2) = \theta^{2,1}(x^1) + \theta^{2,2}(x^1, x^2)$$

where for each player $\nu \in \{1, 2\}$, $\theta^{\nu,1} : \mathbb{R} \to \mathbb{R}$ is a convex and piecewise C^2 function and $\theta^{\nu,2} : \mathbb{R}^2 \to \mathbb{R}$ is a convex and C^2 function.

We suppose that the constraint set-valued mappings of the players are given by

$$X^1(x^2) = \{x^1 \in \mathbb{R} \mid 0 \leq x^1 \leq M^1 \,,\, g(x^1, x^2) \leq 0\}$$

$$X^2(x^1) = \{x^2 \in \mathbb{R} \mid 0 \leq x^2 \leq M^2 \,,\, g(x^1, x^2) \leq 0\},$$

which implies that each player solves the optimization problem:

$$\text{Player 1: } \min_{x^1} \theta^{1,1}(x^1) + \theta^{1,2}(x^1, x^2) \text{ subject to } 0 \leq x^1 \leq M^1 \text{ and } g(x^1, x^2) \leq 0$$

$$\text{Player 2: } \min_{x^2} \theta^{2,1}(x^2) + \theta^{2,2}(x^1, x^2) \text{ subject to } 0 \leq x^2 \leq M^2 \text{ and } g(x^1, x^2) \leq 0.$$

This family of two-player games covers the Cournot duopoly games (which is a Cournot oligopoly game with two firms). Actually, if $\theta^{\nu,1}(x^\nu) = C_\nu(x^\nu)$, where C_ν is the cost function of firm ν, and $\theta^{\nu,2}(x^\nu, x^{-\nu}) = -x^\nu p\left(\sum_{\mu=1}^2 x^\mu\right)$, where p is the unit price function, then we obtain an Cournot duopoly game. The function g can represent a limitation of resources to which each firm is subjected. For more information about the Cournot oligopoly games, see e.g. [11–13]. In [14], the authors use the same tools of variational analysis and generalized differentiation as us in order to derive optimality conditions for some related problems on Cournot–Nash equilibrium.

We suppose that g is a symmetric function in the following sense: for all $(x_1, x_2) \in \mathbb{R}^2, g(x_1, x_2) = g(x_2, x_1)$. This means that in the case of Cournot duopoly games, each firm is affected in the same way by the limitation of resources and implies that

$$\frac{\partial g}{\partial x^1}\left(x^1, x^2\right) = \frac{\partial g}{\partial x^2}\left(x^1, x^2\right). \tag{40}$$

We suppose that for each player $\nu \in \{1,2\}$, there exists a finite set $D(\nu)$ such that $\theta^{\nu,1}$ is C^2 on $\mathbb{R} \setminus D(\nu)$. For more clarity in what follows, we introduce the notation $D(\nu) = \{a_i^\nu \mid i = 1, \dots, q^\nu\}$, $b_i^{\nu,-} = (\theta^{\nu,1})'(a_i^{\nu-})$ and $b_i^{\nu,+} = (\theta^{\nu,1})'(a_i^{\nu+})$, where for a convex function $f : \mathbb{R} \to \mathbb{R}, f'(x^+)$ (resp. $f'(x^-)$) means $\lim_{h\to 0^+} f'(x+h)$ (resp. $\lim_{h\to 0^-} f'(x+h)$). We can observe that if $\overline{x}^\nu \in D(\nu)$, then $\partial\theta^{\nu,1}(\overline{x}^\nu) = [b_i^{\nu,-}, b_i^{\nu,+}]$ where $i \in \{1, \dots, q^\nu\}$ satisfies $\overline{x}^\nu = a_i^\nu$.

For each player $\nu \in \{1,2\}$, for each $i \in \{1, \dots, q^\nu\}$, we introduce the notation $d_i^{\nu+}$ (resp. $d_i^{\nu-}$) defined as follows

$$d_i^{\nu+} = \lim_{h\to 0^+} (\theta^\nu)''(a_i^\nu + h) \quad (\text{resp. } d_i^{\nu-} = \lim_{h\to 0^-} (\theta^\nu)''(a_i^\nu + h))$$

We define the functions

$$g^1(x) = \begin{pmatrix} -x^1 \\ x^1 - M^1 \\ g(x) \end{pmatrix}, \; g^2(x) = \begin{pmatrix} -x^2 \\ x^2 - M^2 \\ g(x) \end{pmatrix},$$

and

$$\tilde{g}_1(x) = -x^1,\ \tilde{g}_2(x) = x^1 - M^1,\ \tilde{g}_3(x) = g(x),\ \tilde{g}_4(x) = -x^2,\ \tilde{g}_5(x) = x^2 - M^2,\ \tilde{g}_6(x) = g(x).$$

Each player solves the following optimization program

$$\text{Player 1: } \min_{x^1} \theta^{1,1}(x^1) + \theta^{1,2}(x^1, x^2) \text{ subject to } g^1(x) \leq 0$$

$$\text{Player 2: } \min_{x^2} \theta^{2,1}(x^2) + \theta^{2,2}(x^1, x^2) \text{ subject to } g^2(x) \leq 0$$

We define $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ as in (16), $G : \mathbb{R}^2 \times \mathbb{R}^6 \to \mathbb{R}^2$ as in (17), $\Phi : \mathbb{R}^2 \times \mathbb{R}^6 \rightrightarrows \mathbb{R}^2 \times \mathbb{R}^6$ as in (4).

Let us define the set-valued mapping $F_1 : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by

$$F_1(x) = \partial\theta^{1,1}(x^1) \times \partial\theta^{2,1}(x^2)$$

and the function $F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_2(x) = \begin{pmatrix} \frac{\partial\theta^{1,2}}{\partial x^1}(x^1, x^2) \\ \frac{\partial\theta^{2,2}}{\partial x^2}(x^1, x^2) \end{pmatrix}$$

Observe that $F = F_1 + F_2$. The following lemma gives a calculus rule for $D^*F_1(\bar{x}|\bar{y})(y^*)$ for an arbitrary $\bar{x}$.

Lemma 6.1: *Let $\bar{x} \in \mathbb{R}^2$ and $\bar{y} \in F_1(\bar{x})$. We have*

$$D^*F_1(\bar{x}|\bar{y})(y^*) = c^1(\bar{x}^1, \bar{y}^1, y^{*,1}) \times c^2(\bar{x}^2, \bar{y}^2, y^{*,2})$$

where for any $\nu \in \{1,2\}$, $c^\nu(\bar{x}^\nu, \bar{y}^\nu, y^{*,\nu})$ is defined as follows: suppose $\bar{x}^\nu \in D(\nu)$, let $i \in \{1,\dots,q^\nu\}$ such that $\bar{x}^\nu = a_i^\nu$; then,

$$c^\nu(\bar{x}^\nu, \bar{y}^\nu, y^{*,\nu}) = \begin{cases} \emptyset & \text{if } b_i^{\nu-} < \bar{y}^\nu < b_i^{\nu+} \text{ and } y^{*,\nu} \neq 0 \\ \mathbb{R} & \text{if } \quad y^{*,\nu} = 0 \\ [d_i^{\nu-} y^{*,\nu}, +\infty[& \text{if } \quad \bar{y}^\nu = b_i^{\nu-} \quad \text{and } y^{\nu,*} > 0 \\ \{d_i^{\nu-} y^{*,\nu}\} & \text{if } \quad \bar{y}^\nu = b_i^{\nu-} \quad \text{and } y^{\nu,*} < 0 \\ ]-\infty, d_i^{\nu+} y^{*,\nu}] & \text{if } \quad \bar{y}^\nu = b_i^{\nu+} \quad \text{and } y^{\nu,*} < 0 \\ \{d_i^{\nu+} y^{*,\nu}\} & \text{if } \quad \bar{y}^\nu = b_i^{\nu+} \quad \text{and } y^{\nu,*} > 0 \end{cases}$$

If $\bar{x}^\nu \notin D(\nu)$, then $c^\nu(\bar{x}^\nu, \bar{y}^\nu, y^{*,\nu}) = \{(\theta^{\nu,1})''(\bar{x}^\nu) y^{*,\nu}\}$.

Proof: See Appendix 1. □

The following proposition gives sufficient conditions for the metric regularity of Φ for this two-player game when the constraint $g(x) \le 0$ is not active. First, we introduce the following notations: for an arbitrary $\bar{x} \in \mathbb{R}^2$ feasible point for GNEP, we write the matrix $JF_2(\bar{x})^\top$ as

$$JF_2(\bar{x})^\top := E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$$

and also define $E_1 = (E_{11}, E_{12})$ and $E_2 = (E_{21}, E_{22})$.

Observe that if $0 < \bar{x}^1 < M^1$, $0 < \bar{x}^2 < M^2$ and $g(\bar{x}) < 0$, then $\bar{x}$ is a solution of GNEP if and only if $0 \in F_1(\bar{x}) + F_2(\bar{x})$, which means that $-\nabla_{x^1}\theta^{1,2}(\bar{x}^1, \bar{x}^2) \in \partial\theta^{1,1}(\bar{x}^1)$ and $-\nabla_{x^2}\theta^{2,2}(\bar{x}^1, \bar{x}^2) \in \partial\theta^{2,1}(\bar{x}^2)$. In the case where $\bar{x}^1 = a_i^1$ with $i \in \{1,\dots,q^1\}$ (resp. $\bar{x}^2 = a_j^2$ with $j \in \{1,\dots,q^2\}$), this implies that $b_i^{1-} \le -\nabla_{x^1}\theta^{1,2}(\bar{x}^1, \bar{x}^2) \le b_i^{1+}$ (resp. $b_j^{2-} \le -\nabla_{x^2}\theta^{2,2}(\bar{x}^1, \bar{x}^2) \le b_j^{2+}$). For each player ν, we define d^ν as follows:

* If $\bar{x}^\nu \notin D(\nu)$, then $d^\nu = (\theta^\nu)''(\bar{x}^\nu)$
* If $\bar{x}^\nu \in D(\nu)$, $\bar{x}^\nu = a_\nu^1$ with $i \in \{1,\dots,q^\nu\}$ and $b_i^{\nu-} < -\nabla_{x^1}\theta^{1,2}(\bar{x}^\nu, \bar{x}^\nu) < b_i^{\nu+}$, then $d^\nu = +\infty$.
* If $\bar{x}^\nu \in D(\nu)$, $\bar{x}^\nu = a_\nu^1$ with $i \in \{1,\dots,q^\nu\}$ and $b_i^{\nu-} = -\nabla_{x^1}\theta^{1,2}(\bar{x}^\nu, \bar{x}^\nu)$, then $d^\nu = d_i^{\nu-}$.
* If $\bar{x}^\nu \in D(\nu)$, $\bar{x}^\nu = a_\nu^1$ with $i \in \{1,\dots,q^\nu\}$ and $-\nabla_{x^1}\theta^{1,2}(\bar{x}^\nu, \bar{x}^\nu) = b_i^{\nu+}$, then $d^\nu = d_i^{\nu+}$.

Proposition 6.2: *Let $\bar{x}$ be a solution of GNEP with $0 < \bar{x}_1 < M_1$ and $0 < \bar{x}_2 < M_2$ and $g(\bar{x}) < 0$. Suppose that $\bar{x}^1 \notin D(1)$ or $\bar{x}^2 \notin D(2)$. If $E_{11} + d^1 \neq 0$, $E_{22} + d^2 \neq 0$, $(E_{11} + d^1)(E_{22} + d^2) > E_{12}E_{21}$, then Φ is metrically regular at $(\bar{x}, 0_6)$.*

Proof: It is a direct consequence of Proposition A.1 given in the Appendix 1. □

The following proposition gives a sufficient condition in the case where $g(\bar{x}) = 0$. For $(\bar{x}, \bar{\lambda}) \in \mathbb{R}^2 \times \mathbb{R}^6$, we use the notations $B = \frac{\partial g}{\partial x_1}(\bar{x}) = \frac{\partial g}{\partial x_2}(\bar{x})$, and

$$E = JF_2(\bar{x})^\top + J_x G(\bar{x}, \bar{\lambda})^\top = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}.$$

For a better understanding, in the following proposition when $(\bar{x}, \bar{\lambda})$ is a solution of the KKT system associated with GNEP and $0 < \bar{x}^1 < M^1$, $0 < \bar{x}^2 < M^2$, $g(\bar{x}) = 0$, we denote by $\bar{\lambda}^1$ the Lagrange multiplier associated with the constraint $\tilde{g}_3(\bar{x}) = 0$, and by $\bar{\lambda}^2$ the Lagrange multiplier associated with the constraint $\tilde{g}_6(\bar{x}) = 0$. Therefore, since the other constraints are not active, we have $\bar{\lambda} = (0, 0, \bar{\lambda}^1, 0, 0, \bar{\lambda}^2)$, $-\nabla_{x^1}\theta^{1,2}(\bar{x}) - B\bar{\lambda}^1 \in \partial\theta^{1,1}(\bar{x})$ and $-\nabla_{x^2}\theta^{2,2}(\bar{x}) - B\bar{\lambda}^2 \in \partial\theta^{2,1}(\bar{x})$. For each player ν, we define d^ν as follows:

* If $\bar{x}^\nu \notin D(\nu)$, then $d^\nu = (\theta^\nu)''(\bar{x}^\nu)$
* If $\bar{x}^\nu \in D(\nu)$, $\bar{x}^\nu = a^1_\nu$ with $i \in \{1, \dots, q^\nu\}$ and $b_i^{\nu-} < -\nabla_{x^1}\theta^{1,2}(\bar{x}^\nu, \bar{x}^\nu) - B\bar{\lambda}^\nu < b_i^{\nu+}$, then $d^\nu = +\infty$.
* If $\bar{x}^\nu \in D(\nu)$, $\bar{x}^\nu = a^1_\nu$ with $i \in \{1, \dots, q^\nu\}$ and $b_i^{\nu-} = -\nabla_{x^1}\theta^{1,2}(\bar{x}^\nu, \bar{x}^\nu) - B\bar{\lambda}^\nu$, then $d^\nu = d_i^{\nu-}$.
* If $\bar{x}^\nu \in D(\nu)$, $\bar{x}^\nu = a^1_\nu$ with $i \in \{1, \dots, q^\nu\}$ and $-\nabla_{x^1}\theta^{1,2}(\bar{x}^\nu, \bar{x}^\nu) - B\bar{\lambda}^\nu = b_i^{\nu+}$, then $d^\nu = d_i^{\nu+}$.

The next proposition gives a sufficient condition for metric subregularity of Φ at $\bar{z} = (\bar{x}, \bar{\lambda})$; just to precise, when (x, λ) is in a neighbourhood of $(\bar{x}, \bar{\lambda})$, we have:

$$\Phi(x, \lambda) = \begin{pmatrix} \partial_{x^1}\theta^1(x) \times \partial_{x^2}\theta^2(x) + G(x, \lambda) \\ \lambda_1 \\ \lambda_2 \\ \min\{-g(x), \lambda_3\} \\ \lambda_4 \\ \lambda_5 \\ \min\{-g(x), \lambda_6\} \end{pmatrix}.$$

Proposition 6.3: *Let $\bar{x}$ be a solution of GNEP with $0 < \bar{x}_1 < M_1$ and $0 < \bar{x}_2 < M_2$ and $g(\bar{x}) = 0$. Let $(0, 0, \bar{\lambda}^1, 0, 0, \bar{\lambda}^2) \in \Lambda(\bar{x})$ be Lagrange multipliers associated with the solution $\bar{x}$ of GNEP. We suppose that $B \neq 0$. If one of the following assumptions holds, then Φ is metrically subregular at $(\bar{x}, 0, 0, \bar{\lambda}^1, 0, 0, \bar{\lambda}^2, 0)$.*

(1) $\bar{x}^1 \notin D(1)$, $\bar{\lambda}^1 > 0$ *and* $E_{22} + d^2 > E_{12}$.
(2) $\bar{x}^2 \notin D(2)$, $\bar{\lambda}^2 > 0$ *and* $E_{11} + d^1 > E_{21}$

Proof: It is a direct consequence of Proposition A.2 given in Appendix 1. □

We apply these results to a Cournot game with two firms. In this Cournot game, each firm solves the following GNEP:

$$\text{Firm 1: } \min_{x^1} c_1(x^1) - x^1 p(x^1 + x^2) \text{ subject to } 0 \le x^1 \le M^1 \text{ and } g(x^1, x^2) \le 0$$

$$\text{Firm 2: } \min_{x^2} c_2(x^2) - x^2 p(x^1 + x^2) \text{ subject to } 0 \le x^2 \le M^2 \text{ and } g(x^1, x^2) \le 0.$$

We suppose that the price function p is defined as the inverse of a linear demand curve:

$$p(y) = \max(\alpha - \beta y, 0)$$

and the function $g(x^1, x^2)$ is the restriction of resources defined as

$$g(x^1, x^2) = \mu(x^1 + x^2) - M$$

where M is the quantity of available resources and μy is the quantity of resources needed for producing quantity y. Naturally, we suppose that $\alpha, \beta, \mu, M > 0$. Moreover, we suppose that $\frac{M}{\mu} < \frac{\alpha}{\beta}$, which ensures that p is differentiable at $\bar{x}^1 + \bar{x}^2$ if $g(\bar{x}^1 + \bar{x}^2) \le 0$.

Using the same notation as before, we have

$$F(x^1, x^2) = \begin{pmatrix} \partial c_1(x^1) - x^1 p'(x^1 + x^2) - p(x^1 + x^2) \\ \partial c_2(x^1) - x^2 p'(x^1 + x^2) - p(x^1 + x^2) \end{pmatrix}.$$

We suppose that the functions c_i are convex, which implies that the functions $x_i \mapsto c_i(x^i) - x^i p(x^1 + x^2)$, $i \in \{1,2\}$, are convex.

Theorem 6.4: *Let $\bar{x}$ be a solution of the Cournot game which satisfies $0 < \bar{x}^1 < M^1$ and $0 < \bar{x}^2 < M^2$ and $\bar{\lambda} := (\bar{\lambda}_i)_{i\in\{1,\dots,6\}}$ a vector of associated Lagrange multipliers. Suppose that $p'(\bar{x}^1 + \bar{x}^2) < 0$ and $p''(\bar{x}^1 + \bar{x}^2) < 0$. If $\bar{x}^1 \notin D(1)$ and $-g(\bar{x}) + \bar{\lambda}_3 > 0$, or $\bar{x}^2 \notin D(2)$ and $-g(\bar{x}) + \bar{\lambda}_6 > 0$, then Φ is metrically subregular at $(0, \bar{z})$, where $\bar{z} = (\bar{x}, \bar{\lambda})$.*

Proof: We first prove the case where $g(\bar{x}) < 0$. We write $F(x^1, x^2) = F_1(x^1, x^2) + F_2(x^1, x^2)$ where $F_1(x^1, x^2) = \partial c_1(x^1) \times \partial c_2(x^2)$ and

$$F_2(x^1, x^2) = \begin{pmatrix} -x^1 p'(x^1 + x^2) - p(x^1 + x^2) \\ -x^2 p'(x^1 + x^2) - p(x^1 + x^2) \end{pmatrix}.$$

Let $E = JF_2^\top(\bar{x})$ and (d^1, d^2) are defined in the same way than it has been defined before Proposition 6.2. Observe that $d^1 \geq 0$ and $d^2 \geq 0$ because c_1 and c_2 are convex functions. By Proposition 6.2, if $(d_1 + E_{11})(d_2 + E_{22}) > E_{12}E_{21}$, then Φ is metrically subregular at $(0, \bar{z})$. Actually, we have

$$E = \begin{pmatrix} 2\beta & \beta \\ \beta & 2\beta \end{pmatrix}$$

then

$$\begin{aligned}(d_1 + E_{11})(d_2 + E_{22}) - E_{12}E_{21} &= (d_1 + 2\beta)(d_2 + 2\beta) - \beta^2 \\ &= d_1 d_2 + 2\beta(d_1 + d_2) + 3\beta^2 \\ &> 0\end{aligned}$$

We now consider the case where $g(\bar{x}) = 0$. In this case, we suppose without loss of generality that $\bar{x}^1 \notin D(1)$ and $\bar{\lambda}_3 > 0$. By Proposition 6.3, it is sufficient to prove that $E_{22} + d^2 > E_{21}$, where E is given by $E = JF_2(\bar{x})^\top + J_x G(\bar{x}, \bar{\lambda})^\top$ and (d^1, d^2) is defined as before Proposition 6.3. Since

$$G(x, \bar{\lambda}) = \begin{pmatrix} \frac{\partial g}{\partial x_1}(x)\bar{\lambda}_3 \\ \frac{\partial g}{\partial x_2}(x)\bar{\lambda}_6 \end{pmatrix} = \begin{pmatrix} \mu\bar{\lambda}_3 \\ \mu\bar{\lambda}_6 \end{pmatrix}$$

one has

$$J_x G(\bar{x}, \bar{\lambda}) = 0.$$

Finally:

$$E = \begin{pmatrix} 2\beta & \beta \\ \beta & 2\beta \end{pmatrix}$$

Then, we have

$$E_{22} + d^2 - E_{21} = \beta + d^2 > 0$$

Therefore, Φ is metrically subregular at $(\bar{x}, 0, 0, \bar{\lambda}_3, 0, 0, \bar{\lambda}_6)$. □

We can observe that these propositions do not consider the case where $\bar{x}^1 \in D(1)$ and $\bar{x}^2 \in D(2)$. Actually, in this case, the sufficient conditions of Theorems 4.3 and 4.4 are not satisfied.

7. Conclusion

In this article, we have studied the metric subregularity and the stability in GNEP with non-smooth loss functions. In order to achieve this goal, we have used the coderivative criterion of Mordukhovich which characterizes the metric regularity of set-valued mapping. Nevertheless, the conclusion of Theorem 4.4 is that Φ is metrically subregular and not metrically regular (in Example 4.5, we show that the hypotheses of Theorem 4.4 do not ensure the metric regularity of Φ). This illustrates that the assumptions of Theorem 4.4 are not so strong. Moreover, despite the fact that the adapted tool for metric subregularity is the outer-coderivative (see e.g. [19]), we can prove that if one or more constraints are not active, the coderivative of Φ is equal to the outer-coderivative of Φ, which proves that making the assumptions of Theorems 4.3 and 4.4 weaker will not be easy.

At the same time, in Section 6, we could not apply Theorems 4.3 or 4.4 when $\bar{x}^1 \in D(1)$ and $\bar{x}^2 \in D(2)$, while in many of these cases, Φ is metrically regular or subregular. This is a limit in the potential application of these theorems, thus replacing the coderivative D^*F with another tool of nonsmooth analysis in order that Theorems 4.3 and 4.4 can be applied in the case when $\bar{x}^1 \in D(1)$ and $\bar{x}^2 \in D(2)$ could be a possible extension of this work. Another natural extension of this work would be to consider the case where the strict complementarity assumption is violated.

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Appendix 1.

In this Appendix 1, we use the same notation as in Section 6.

Proof of Lemma 6.1: We first observe that $\mathrm{Gr}(F_1) = \{(x^1, x^2, y^1, y^2) \mid (x^1, y^1, x^2, y^2) \in \mathrm{Gr}(\partial\theta^{1,1}) \times \mathrm{Gr}(\partial\theta^{2,1})\}$ and deduce that $D^*F_1(\bar{x}|\bar{y})(y^*) = D^*(\partial\theta^{1,1})(\bar{x}^1|\bar{y}^1)(y^{*,1}) \times D^*(\partial\theta^{2,1})(\bar{x}^2|\bar{y}^2)(y^{*,2})$. We only compute $D^*(\partial\theta^{1,1})(\bar{x}^1|\bar{y}^1)$ $(y^{*,1})$ since $D^*(\partial\theta^{2,1})(\bar{x}^2|\bar{y}^2)(y^{*,2})$ can be computed in the same way.

If $\bar{x}^1 \notin D(1)$, $\theta^{1,1}$ is C^2 at $\bar{x}^1$, then $D^*(\partial\theta^{1,1})(\bar{x}^1|\bar{y}^1)(y^{*,1}) = \{(\theta^{1,1})''(\bar{x}^1)y^{*,1}\}$ by Proposition 3.1. We now suppose that $\bar{x}^1 \in D(1)$. Let $i \in \{1,\dots,q^\nu\}$ such that $\bar{x}^\nu = a_i^\nu$.

Since $\partial\theta^{1,1}(\bar{x}^\nu) = [b_i^{\nu-}, b_i^{\nu+}]$, we have $b_i^{\nu-} \le \bar{y}_i^\nu \le b_i^{\nu+}$. We consider three cases.

Case 1: $b_i^{\nu-} < \bar{y}^\nu < b_i^{\nu+}$. In this case, $N_L((\bar{x}^1,\bar{y}^1), \mathrm{Gr}(\partial\theta^{1,1})) = \mathbb{R}\times\{0\}$, then $D^*(\partial\theta^{1,1})(\bar{x}^1|\bar{y}^1)(y^{*,1}) = \emptyset$ if $y^{*,1} \neq 0$ and $D^*(\partial\theta^{1,1})(\bar{x}^1|\bar{y}^1)(y^{*,1}) = \mathbb{R}$ if $y^{*,1} = 0$.

Case 2: $\bar{y}^\nu = b_i^{\nu-}$. We can observe that

$$N_L((\bar{x}^1,\bar{y}^1), \mathrm{Gr}(\partial\theta^{1,1})) = \mathbb{R}\times\{0\} \cup \mathbb{R}(d_i^{1-},-1) \cup \left(\mathbb{R}_+(1,0) + \mathbb{R}_+(d_i^{1-},-1)\right).$$

From this description of $N_L((\bar{x}^1,\bar{y}^1), \mathrm{Gr}(\partial\theta^{1,1}))$, we can deduce the expression of $D^*(\partial\theta^{1,1})(\bar{x}^1|\bar{y}^1)(y^{*,1})$.

Case 3: $\bar{y}^\nu = b_i^{\nu+}$. This case can be treated the same way as case 2. □

Proposition A.1: *Let $\bar{x}$ a solution of GNEP with $0 < \bar{x}_1 < M_1$ and $0 < \bar{x}_2 < M_2$ and $g(\bar{x}) < 0$. If one of the following assumptions holds, with $E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22}\end{pmatrix} := JF_2(\bar{x})^\top$, then Φ is metrically regular around $(\bar{x}, 0_6, 0)$, where $0_6 = (0,0,0,0,0,0)$.*

(1) $\bar{x}^1 \notin D(1)$, $\bar{x}^2 \notin D(2)$ *and the matrix* $E + \begin{pmatrix} (\theta^{1,1})''(\bar{x}^1) & 0 \\ 0 & (\theta^{2,1})''(\bar{x}^2)\end{pmatrix}$ *is nonsingular.*

(2) $\bar{x}^1 = a_i^1$, *with* $i \in \{1,\dots,q^1\}$, $\bar{x}^2 \notin D(2)$, $b_i^{1-} < -\nabla_{x^1}\theta^{1,2}(\bar{x}^1,\bar{x}^2) < b_i^{1+}$ *and* $(\theta^{21})''(\bar{x}^2) + E_{22} \neq 0$.

(3) $\bar{x}^1 = a_i^1$, *with* $i \in \{1,\dots,q^1\}$, $\bar{x}^2 \notin D(2)$, $b_i^{1-} = -\nabla_{x^1}\theta^{1,2}(\bar{x}^1,\bar{x}^2)$, $(\theta^{21})''(\bar{x}^2) + E_{22} \neq 0$ *and*

$$E_{11} + d_i^{1-} > \frac{E_{12}E_{21}}{E_{22} + (\theta^{21})''(\bar{x}^2)}$$

(4) $\bar{x}^1 = a_i^1$, *with* $i \in \{1,\dots,q^1\}$, $\bar{x}^2 \notin D(2)$, $b_i^{1+} = -\nabla_{x^1}\theta^{1,2}(\bar{x}^1,\bar{x}^2)$, $(\theta^{21})''(\bar{x}^2) + E_{22} \neq 0$ *and*

$$E_{11} + d_i^{1+} > \frac{E_{12}E_{21}}{E_{22} + (\theta^{21})''(\bar{x}^2)}$$

(5) $\bar{x}^1 \notin D(1)$, $\bar{x}^2 = a_i^2$, *with* $i \in \{1,\dots,q^2\}$, $b_i^{2-} < -\nabla_{x^2}\theta^{2,2}(\bar{x}^1,\bar{x}^2) < b_i^{2+}$ *and* $(\theta^{11})''(\bar{x}^1) + E_{11} \neq 0$.

(6) $\bar{x}^1 \notin D(1)$, $\bar{x}^2 = a_i^2$, *with* $i \in \{1,\dots,q^2\}$, $b_i^{2-} = -\nabla_{x^2}\theta^{2,2}(\bar{x}^1,\bar{x}^2)$, $(\theta^{11})''(\bar{x}^1) + E_{11} \neq 0$ *and*

$$E_{22} + d_i^{2-} > \frac{E_{12}E_{21}}{E_{11} + (\theta^{11})''(\bar{x}^1)}$$

(7) $\bar{x}^1 \notin D(1)$, $\bar{x}^2 = a_i^2$, *with* $i \in \{1,\dots,q^2\}$, $b_i^{2+} = -\nabla_{x^2}\theta^{2,2}(\bar{x}^1,\bar{x}^2)$, $(\theta^{11})''(\bar{x}^1) + E_{11} \neq 0$ *and*

$$E_{22} + d_i^{2+} > \frac{E_{12}E_{21}}{E_{11} + (\theta^{11})''(\bar{x}^1)}$$

(8) $\bar{x}^1 = a_{i_1}^1$, $\bar{x}^2 = a_{i_2}^2$, *with* $i_1 \in \{1,\dots,q^1\}$ *and* $i_2 \in \{1,\dots,q^2\}$, $b_i^{1+} = -\nabla_{x^1}\theta^{1,2}(\bar{x}^1,\bar{x}^2)$, $b_i^{2+} = -\nabla_{x^2}\theta^{2,2}(\bar{x}^1,\bar{x}^2)$, $d_{i_1}^{1+} + E_{11} \neq 0$, $d_{i_2}^{2+} + E_{22} \neq 0$ *and*

$$(d_{i_1}^{1+} + E_{11})(d_{i_2}^{2+} + E_{22}) > E_{12}E_{21}$$

Proof: We will prove items 1–3; the other items can be proved the same way. We use the sufficient condition of Theorem 4.3. We can observe that $A = \emptyset$, where $A = A(\bar{x})$ has been defined in (11). Since $\bar{\lambda} = 0_6$, the assumption of Theorem 4.3 can be written as

$$\begin{cases} 0 \in c^1(\bar{x}^1, -\nabla_{x^1}\theta^{1,2}(\bar{x}^1,\bar{x}^2), y^{*,1}) + E_1 y^* \\ 0 \in c^2(\bar{x}^2, -\nabla_{x^2}\theta^{2,2}(\bar{x}^1,\bar{x}^2), y^{*,2}) + E_2 y^* \end{cases} \Longrightarrow y^* = 0$$

Then, we consider a vector y^* which satisfies

$$0 \in c^1(\bar{x}^1, -\nabla_{x^1}\theta^{1,2}(\bar{x}^1,\bar{x}^2), y^{*,1}) + E_1 y^*\ ,\ 0 \in c^2(\bar{x}^2, -\nabla_{x^2}\theta^{2,2}(\bar{x}^1,\bar{x}^2), y^{*,2}) + E_2 y^*$$

and prove that $y^* = 0$ in each case.

Case where assumption 1 is satisfied: In this case, using Lemma 6.1, we obtain that

$$\left(E+\begin{pmatrix}(\theta^{1,1})''(\bar{x}^1) & 0\\ 0 & (\theta^{2,1})''(\bar{x}^2)\end{pmatrix}\right)\begin{pmatrix}y^{*,1}\\ y^{*,2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

By assumption, the above matrix is non-singular, which implies that $y^*=0$.
Case where assumption 2 is satisfied: Since $c^1(\bar{x}^1,-\nabla_{x^1}\theta^{1,2}(\bar{x}),y^{*,1})\neq\emptyset$ and $b_i^{1-}<-\nabla_{x^1}\theta^{1,2}(\bar{x}^1,\bar{x}^2)<b_i^{1+}$, by Lemma 6.1, we have , $y^{*,1}=0$; then, the second equation becomes $((\theta^{2,1})''(\bar{x}^2)+E_{22})y^{*,2}=0$ which implies that $y^{*,2}=0$ since $(\theta^{2,1})''(\bar{x}^2)+E_{22}\neq 0$.
Case where assumption 3 is satisfied: In the same way as the previous case, $y^{*,1}=0\Rightarrow y^{*,2}=0$. We suppose that $y^{*,1}<0$; then, by Lemma 6.1, we have $c^1(\bar{x}^1,-\nabla_{x^1}\theta^{1,2}(\bar{x}),y^{*,1})=\{d_i^{1-}y^{*,1}\}$; thus, we obtain

$$\left(E+\begin{pmatrix}d_i^{1-} & 0\\ 0 & (\theta^{2,1})''(\bar{x}^2)\end{pmatrix}\right)y^*=\begin{pmatrix}0\\0\end{pmatrix}$$

This matrix, which can be written as $\begin{pmatrix}E_{11}+d_i^{1-} & E_{12}\\ E_{21} & E_{22}+(\theta^{21})''(\bar{x}^2)\end{pmatrix}$, is non-singular by the assumption $E_{11}+d_i^{1-}>\dfrac{E_{12}E_{21}}{E_{22}+(\theta^{21})''(\bar{x}^2)}$, which contradicts $y^{*,1}<0$.

We now suppose that $y^{*,1}>0$. In this case, by Lemma 6.1, we have $c^1(\bar{x}^1,-\nabla_{x^1}\theta^{1,2}(\bar{x}),y^{*,1})=[d_i^{1-}y^{*,1},+\infty[$. Thus, the system

$$\begin{cases}0\in c^1(\bar{x}^1,-\nabla_{x^1}\theta^{1,2}(\bar{x}),y^{*,1})+E_1y^*\\ (\theta^{2,1})''(\bar{x}^2)y^{*,2}+E_2y^*=0\end{cases}$$

can be written as

$$\begin{cases}d_i^{1-}y^{*,1}+E_1y^*\leq 0\\ (\theta^{2,1})''(\bar{x}^2)y^{*,2}+E_2y^*=0\end{cases}$$

Consider $m=E_{11}+d_i^{1-}-\dfrac{E_{12}E_{21}}{E_{22}+(\theta^{21})''(\bar{x}^2)}>0$. We have

$$(E_{11}+d_i^{1-},E_{12})=\frac{E_{12}}{E_{22}+(\theta^{21})''(\bar{x}^2)}(E_{21},E_{22}+(\theta^{21})''(\bar{x}^2))+(m,0)$$

which implies that

$$\begin{aligned}d_i^{1-}y^{*,1}+E_1y^* &= (E_{11}+d_i^{1-},E_{12})\begin{pmatrix}y^{*,1}\\ y^{*,2}\end{pmatrix}\\ &=\frac{E_{12}}{E_{22}+(\theta^{21})''(\bar{x}^2)}(E_{21},E_{22}+(\theta^{21})''(\bar{x}^2))\begin{pmatrix}y^{*,1}\\ y^{*,2}\end{pmatrix}+(m,0)\begin{pmatrix}y^{*,1}\\ y^{*,2}\end{pmatrix}\\ &=\frac{E_{12}}{E_{22}+(\theta^{21})''(\bar{x}^2)}((\theta^{21})''(\bar{x}^2)y^{*,2}+E_2y^*)+my^{*,1}\\ &=my^{*,1}\\ &>0\end{aligned}$$

because $m>0$ and $y^{*,1}>0$. That contradicts $d_i^{1-}y^{*,1}+E_1y^*\leq 0$. Finally, we have $y^{*,1}=0$; then, $y^{*,2}=0$.

The cases where assumptions 4–7 are satisfied can be treated the same way. By Theorem 4.3, Φ is metrically regular around $(\bar{x},0_6,0)$. □

Proposition A.2: *Let $\bar{x}$ be a solution of GNEP with $0<\bar{x}_1<M_1$ and $0<\bar{x}_2<M_2$ and $g(\bar{x})=0$. Let $(0,0,\bar{\lambda}^1,0,0,\bar{\lambda}^2)\in\Lambda(\bar{x})$ be Lagrange multipliers associated with the solution $\bar{x}$ of GNEP. We suppose that $B\neq 0$. If one of the following assumptions holds, with $E=\begin{pmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{pmatrix}=JF_2(\bar{x})^\top+J_xG(\bar{x},\bar{\lambda})^\top$ then Φ is metrically subregular at $(\bar{x},0,0,\bar{\lambda}^1,0,0,\bar{\lambda}^2,0)$.*

(1) *$\bar{x}^1\notin D(1)$, $\bar{x}^2\notin D(2)$, $\bar{\lambda}^1>0$ and $E_{11}+(\theta^{11})''(\bar{x}^2)\neq E_{21}$, or, $\bar{\lambda}^2>0$ and $E_{22}+(\theta^{21})''(\bar{x}^2)\neq E_{12}$.*
(2) *$\bar{x}^1=a_i^1$, with $i\in\{1,\dots,q^1\}$, $\bar{x}^2\notin D(2)$, $\bar{\lambda}^2>0$ and $b_i^{1-}<-\nabla_{x^1}\theta^{1,2}(\bar{x}^1,\bar{x}^2)-B\bar{\lambda}^1<b_i^{1+}$.*
(3) *$\bar{x}^1\notin D(1)$, $\bar{x}^2=a_i^2$, with $i\in\{1,\dots,q^2\}$, $\bar{\lambda}^1>0$ and $b_i^{2-}<-\nabla_{x^2}\theta^{2,2}(\bar{x}^1,\bar{x}^2)-B\bar{\lambda}^2<b_i^{2+}$.*
(4) *$\bar{x}^1=a_i^1$, with $i\in\{1,\dots,q^1\}$, $\bar{x}^2\notin D(2)$, $\bar{\lambda}^2>0$, $b_i^{1-}=-\nabla_{x^1}\theta^{1,2}(\bar{x}^1,\bar{x}^2)-\bar{\lambda}^1B$ and*

$$E_{11}+d_i^{1-}>E_{21}$$

(5) $\bar{x}^1 = a_i^1$, *with* $i \in \{1,\dots,q^1\}$, $\bar{x}^2 \notin D(2)$, $\bar{\lambda}^2 > 0$, $b_i^{1+} = -\nabla_{x^1}\theta^{1,2}(\bar{x}^1,\bar{x}^2) - \bar{\lambda}^1 B$ *and*

$$E_{11} + d_i^{1+} > E_{21}$$

(6) $\bar{x}^1 \notin D(1)$, $\bar{x}^2 = a_i^2$, *with* $i \in \{1,\dots,q^2\}$, $\bar{\lambda}^1 > 0$, $b_i^{2-} = -\nabla_{x^2}\theta^{2,2}(\bar{x}^1,\bar{x}^2) - \bar{\lambda}^2 B$ *and*

$$E_{22} + d_i^{2-} > E_{12}$$

(7) $\bar{x}^1 \notin D(1)$, $\bar{x}^2 = a_i^2$, *with* $i \in \{1,\dots,q^2\}$, $\bar{\lambda}^1 > 0$, $b_i^{2+} = -\nabla_{x^2}\theta^{2,2}(\bar{x}^1,\bar{x}^2) - \bar{\lambda}^2 B$ *and*

$$E_{22} + d_i^{2+} > E_{12}$$

Proof: We use Theorem 5.2. The set $Q(\bar{x})$ defined in (26) is equal to

$$Q(\bar{x}) = \{\{3\},\{6\}\};$$

then, the assumption of Theorem 4.4 can be written as

$$\begin{cases} 0 \in c^1(\bar{x}^1, -\nabla_{x^1}\theta^{1,2}(\bar{x}) - \bar{\lambda}^1 B, y^{*,1}) + E_1 y^* + Bz^* \\ 0 \in c^2(\bar{x}^2, -\nabla_{x^2}\theta^{2,2}(\bar{x}) - \bar{\lambda}^2 B, y^{*,2}) + E_2 y^* + Bz^* \\ By^{*,\beta(\alpha)} = 0 \end{cases} \Longrightarrow y^* = 0\,,\ z^* = 0 \tag{A1}$$

where $\alpha \in Q(\bar{x})$ satisfies $\bar{\lambda}_\alpha > 0$, and $\beta(\alpha) = 1$ if $\alpha = \{3\}$, $\beta(\alpha) = 2$ if $\alpha = \{6\}$. We now prove items 1, 2 and 4.

(1) If $E_{11} + (\theta^{11})''(\bar{x}^2) \neq E_{21}$ and $\bar{\lambda}^2 > 0$, then we choose $\alpha = \{6\}$. The equality $By^{*,2} = 0$ implies $y^{*,2} = 0$ since $B \neq 0$. The system in (A1) can be written as

$$\begin{pmatrix} E_{11} + (\theta^{1,1})''(\bar{x}^1) & B \\ E_{21} & B \end{pmatrix}\begin{pmatrix} y^{*,1} \\ z^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The assumption $E_{11} + (\theta^{11})''(\bar{x}^2) \neq E_{21}$ implies that the above matrix is non-singular, so $y^{*,1} = z^* = 0$.

If $E_{11} + (\theta^{11})''(\bar{x}^2) = E_{21}$, then $E_{22} + (\theta^{21})''(\bar{x}^2) \neq E_{12}$ and $\bar{\lambda}^1 > 0$; then, we can prove item 1 the same way as before considering $\alpha = \{4\}$.

(2) In this case, we consider $\alpha = \{6\}$. Since $By^{*,2} = 0$, we have $y^{*,2} = 0$. Since $c^1(\bar{x}^1, -\nabla_{x^1}\theta^{1,2}(\bar{x}) - \lambda_1 B, y^{*,1}) \neq \emptyset$ and $b_i^{1-} < -\nabla_{x^1}\theta^{1,2}(\bar{x}) - \lambda_1 B < b_i^{1+}$, we deduce by Lemma 6.1 that $y^{*,1} = 0$. Finally, we have $Bz^* = 0$ which implies that $z^* = 0$.

(3) This case can be proved the same way as the previous case.

(4) In this case, we take $\alpha = \{6\}$. Since $By^{*,2} = 0$, we have $y^{*,2} = 0$. Since $-\nabla_{x^1}\theta^{1,2}(\bar{x}) - \lambda_1 B = b_i^{1-}$, by (A1), we have

$$\begin{cases} 0 \in c^1(\bar{x}^1, b_i^{1-}, y^{*,1}) + E_{11}y^{*,1} + Bz^* \\ E_{21}y^{*,1} + Bz^* = 0 \end{cases} \tag{A2}$$

If $y^{*,1} = 0$, then $z^* = 0$ since $B \neq 0$. If $y^{*,1} < 0$, by Lemma 6.1, the system (A2) can be written as

$$\begin{pmatrix} E_{11} + d_i^{1-} & B \\ E_{21} & B \end{pmatrix}\begin{pmatrix} y^{*,1} \\ z^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The assumption $E_{11} + d_i^{1-} > E_{21}$ implies that the above matrix is non-singular; then, $y^{*,1} = 0$, which is a contradiction with $y^{*,1} < 0$. We now suppose that $y^{*,1} > 0$. In this case, by Lemma 6.1, the system (A2) can be written as

$$\begin{cases} 0 \in [d_i^{1-}y^{*,1}, +\infty[+ E_{12}y^{*,1} + Bz^* \\ E_{21}y^{*,1} + Bz^* = 0 \end{cases}$$

which implies that

$$\begin{cases} d_i^{1-}y^{*,1} + E_{11}y^{*,1} + Bz^* \leq 0 \\ E_{21}y^{*,1} + Bz^* = 0 \end{cases}$$

Define $m = E_{11} + d_i^{1-} - E_{21} > 0$; we have $(E_{11} + d_i^{1-}, B) = (E_{21}, B) + (m, 0)$; then,

$$\begin{aligned} d_i^{1-} y^{*,1} + E_{11} y^{*,1} + Bz^* &= (E_{11} + d_i^{1-}, B) \begin{pmatrix} y^{*,1} \\ z^* \end{pmatrix} \\ &= (E_{21}, B) \begin{pmatrix} y^{*,1} \\ z^* \end{pmatrix} + m y^{*,1} \\ &> E_{21} y^{*,1} + Bz^* \text{ since } m > 0 \text{ and } y^{*,1} > 0 \\ &= 0. \end{aligned}$$

We deduce a contradiction with $d_i^{1-} y^{*,1} + E_{11} y^{*,1} + Bz^* \leq 0$, which implies that $y^{*,1} = 0$.

Cases 5, 6 and 7 can be deduced the same way; then, by Theorem 4.4, Φ is metrically subregular at $(\bar{x}, 0, 0, \bar{\lambda}^1, 0, 0, \bar{\lambda}^2, 0)$. $\square$