# ELECTRICITY SPOT MARKET WITH TRANSMISSION LOSSES 

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#### Abstract

In order to study deregulated electricity spot markets, various models have been proposed. Most of them correspond to a, so-called, multi-leader-follower game in which an Independent System Operator (ISO) plays a central role. Our aim in this paper is to consider quadratic bid functions together with the transmission losses in the multi-leader-follower game. Under some reasonable assumptions we deduce qualitative properties for the ISO's problem. In the two islands type market, the explicit formulae for the optimal solutions of the ISO's problem are obtained and we show the existence of an equilibrium.


1. Introduction. With the deregulation and privatization of the electricity market in many countries since the mid 1980's, new models have appeared in the literature. One classical family of economical models for electricity markets is based on Cournot-type formulations, that is a noncooperative game in which the generators compete only with the energy quantities. It is this point of view which will be developed in the present work. Another famous approach is the so-called Supply Function Equilibrium approach (SFE) in which each generator is bidding supply function, that is price-quantity curves. This approach, based on the pioneering work of Klemperer-Meyer [19], has been then applied to electricity markets by many authors and for different purposes; e.g. Green-Newbery[13] for application to the market of England and Wales, Baldick-Hogan [2] for studying stability of the model under perturbations, Anderson-Xu [1] for SFE models under uncertained demand; see also [5, 3, 23]. For qualitative and quantitative comparisons of both approaches the interested reader can refer to [2, 22, 24].

Following the literature about Cournot-type bilevel approaches, the model proposed in this paper can be described as follows: the electricity market is supposed to be centralized by an Independent System Operator (ISO) and each agent bids

[^0]his cost of production to the ISO who computes the best response/dispatch in order to minimize the general cost of production. This leads to a so-called multi-leaderfollower game in which each agent is facing a bilevel problem. Consumers and electricity generators are located in different nodes connected by power transmission lines. In most of the proposed models $[12,18,10,11,16,4]$, the bids of generators are assumed to follow linear or affine laws. Nevertheless it appears interesting to consider spot markets in which the bid functions of the generators (production) are more structured than linear or affine. Namely in [17], Hobbs and Pang used piecewise linear demand functions to obtain a reformulation of the model as a quasivariational inequality while in [15] demand and cost functions were assumed to be quadratic. On the other hand, in some situations as long distance transportation, high intensity (DC transmission), the transmission losses along the lines cannot be neglected. The influence of the transmission losses on the equilibrium has been studied in [11, 8, 4, 14].

Our aim in the work is to consider simultaneously non linear bid functions for the generators and transmission losses on the network. This approach has been used very recently by Henrion-Outrata-Surowiec [14] to propose a reformulation of the spot market as an Equilibrium Problem with Equilibrium constraints, often denoted as EPEC. They provide sharp qualification conditions for this EPEC reformulation.

In the present work we mainly concentrate on the existence of noncooperative equilibrium on the electricity spot market. Uniqueness results, at the equilibrium, for the production and flow along the lines are also proved. The existence of a noncooperative equilibrium is obtained for a special configuration of the network, called the two islands type problem. It corresponds to the case where generators can be separated into groups, virtual or physical islands, linked by one non oriented line (or equivalentely by two oriented lines). This topography fits the situation of the New Zealand market and has been considered, e.g., in [8, 21] and references therein.

The paper is organized as follows. In the next section the general model and hypotheses for the electricity spot market are presented. As in [11, 14], in our model the transmission losses are split between the receiver and the sender. Section 3 is composed of two subsections, the first one being devoted to sufficient conditions ensuring the uniqueness of the solution of the ISO's problem while in the second one, under the assumption of uniqueness of the solution of the ISO's problem, we prove qualitative properties of this solution. Finally, in section 4, we consider the case of the two islands type market and provide an explicit formula for the production vector and the flow vector in terms of the bid of the generators. Then the multi-leader-follower problem modelling the spot market turns out to be a Nash equilibrium problem for which we prove the existence of an equilibrium.
2. Spot market: The mathematical model. We consider a electricity spot market based on a transmission network. The agents of the markets are the producers. Each node of the network is composed of a producer and of a consumer (eventually with a null demand). The lines of the network are supposed to be oriented, each line being split into two oriented lines if the network is not oriented. Each line has a maximal transmission capacity and thermal losses are considered. According to classical models the thermal losses are proportional to the square of the transmission flow along the line.

* $\mathcal{N}$ is the set of nodes ( $N$ being its cardinal).
* $\mathcal{L}$ is the set of electricity lines.
* $L_{i j}$ is the coefficient of thermal losses.

The demand of each agent (as a consumer) is supposed to be given.

* $D_{i}$ is the demand of node $i \in \mathcal{N}$.
* $A_{i} q_{i}+B_{i} q_{i}^{2}$ is the real cost of generation of the node $i \in \mathcal{N}$.


## A - The ISO's PROBLEM

The spot market is regulated by an Independent System Operator (ISO). Thus any agent (as a producer) provides to the ISO a quadratic bid function given by the parameters $a_{i}$ and $b_{i}$. Then the ISO computes the set of admissible productions and transmissions.

* $q_{i} \geq 0$ represents the production at node $i \in \mathcal{N}$.
* $t_{i j}$ represents the flow along the line $i j \in \mathcal{L}$.
* $a_{i} q_{i}+b_{i} q_{i}^{2}$ corresponds to the bid function (bid generation cost) given by producer $i$ to the ISO.
Since the production vector $q$ is supposed to be nonnegative (i.e. $q_{i} \geq 0$, for all $i$ ), the agents (producers) are only acting as producers and not as retailers.

For each line $i j \in \mathcal{L}$ and any flow $t_{i j}$ along this line, the cost $L_{i j} t_{i j}^{2}$ of thermal losses is covered equally by both producer $i$ and producer $j$. This choice of distribution of the thermal losses costs has also been used in Escobar-Jofre [11].

Thus the ISO, knowing the bid vectors $a=\left(a_{1} ; \cdots ; a_{N}\right)$ and $b=\left(b_{1} ; \cdots ; b_{N}\right)$ announced by producers, computes $q=\left(q_{i}\right)_{i \in \mathcal{N}}$ and $t=\left(t_{i j}\right)_{i j \in \mathcal{L}}$ in order to minimize the total generation cost, that is to solve the following optimization problem, denoted by $\operatorname{ISO}(a, b)$

$$
\begin{aligned}
\min _{q, t} & \sum_{i \in \mathcal{N}}\left(a_{i} q_{i}+b_{i} q_{i}^{2}\right) \\
\text { s.t. } & \left\{\begin{array}{l}
q_{i} \geq 0, i=1 \cdots, N \\
q_{i}-\sum_{k: i k \in \mathcal{L}}\left(t_{i k}+\frac{L_{i k}}{2} t_{i k}^{2}\right)+\sum_{k: k i \in \mathcal{L}}\left(t_{k i}-\frac{L_{k i}}{2} t_{k i}^{2}\right) \geq D_{i}, i=1, \cdots, N(b) \\
t_{i j} \geq 0, i j \in \mathcal{L}
\end{array}\right.
\end{aligned}
$$

The only nontrivial assumption of the above problem, assumption (b), simply expresses that the demand at node $i$ is satisfied. Equality in this constraint (b) corresponds to a Kirchhoff law (see Lemma 3.1). The solution set of the above optimization problem will be denoted by $\mathrm{Q}(a, b)$. It is immediate that for any $(a, b) \in\left(\mathbb{R}_{+}^{N}\right)^{2}$, this solution set is nonempty.

Let us observe that, since each $b_{i}$ is nonnegative, the objective function $g_{(a, b)}$ : $q \mapsto \sum_{i \in \mathcal{N}}\left(a_{i} q_{i}+b_{i} q_{i}^{2}\right)$ of the ISO's problem is a convex function. Additionally, the constraint set $K$ of this problem is also clearly convex.

## B - The producer's problem

It is clear that the producers cannot act independently from each other on the market, at least because of the finiteness of the demand. Each producer $i$ aims to maximize his profit, given by the difference between the revenue $\left(a_{i}+2 b_{i} q_{i}\right) q_{i}$ and the real generation cost $A_{i} q_{i}+B_{i} q_{i}^{2}$. But the producer $i$ has to take into account that the production vector $q$ is given by the ISO's problem and therefore depends on the bids of the other producers, namely of the bid vectors $a_{-i}=\left(a_{j}\right)_{j \neq i}$ and
$b_{-i}=\left(b_{j}\right)_{j \neq i}$. Thus the spot electricity market is naturally described as a multi-leader-follower game, with each producer solving the following bilevel optimization problem $P_{i}\left(a_{-i}, b_{-i}\right)$

$$
\begin{aligned}
\max _{a_{i}, b_{i}, q, t} & \left(a_{i}+2 b_{i} q_{i}\right) q_{i}-\left(A_{i} q_{i}+B_{i} q_{i}^{2}\right) \\
\text { s.t. } & \left\{\begin{array}{l}
\underline{A}_{i} \leq a_{i} \leq \bar{A}_{i} \\
\underline{B}_{i} \leq b_{i} \leq \bar{B}_{i} \\
\left(q=\left(q_{i}\right)_{i \in \mathcal{N}}, t=\left(t_{i j}\right)_{i j \in \mathcal{L}}\right) \in \mathrm{Q}(a, b)
\end{array}\right.
\end{aligned}
$$

with $0 \leq \underline{A}_{i} \leq \bar{A}_{i}$ and $0 \leq \underline{B}_{i} \leq \bar{B}_{i}$.
Let us observe that, since $b_{i}$ is positive, the price $a_{i}+2 b_{i} q_{i}$ is increasing whenever the supply $q_{i}$ is increasing. It corresponds to situation in which the "base offer" is made thanks to low cost energy plants while when the supply is increasing some more costly plants have to be activated.

Here the chosen point of view is the optimistic formulation since maximum is taken with regards to variables $(q, t)$. This choice of formulation doesn't have influence in the sequel since our assumptions (see e.g. Proposition 2 and Proposition 4) will ensure uniqueness of the solution of the ISO's problem.

In the above optimization problem, producer $i$ is only concerned with the production value $q_{i}$. But he can only choose a production $q_{i}$ for which there exists a production vector of the other agents $q_{-i}=\left(q_{j}\right)_{j \neq i}$ and a vector of flows $t=\left(t_{i j}\right)_{i j \in \mathcal{L}}$ such that $\left(q_{i}, q_{-i}, t\right) \in \mathrm{Q}(a, b)$. Thus an equilibrium for the (global) spot market is a vector of bids $\left(a^{*}, b^{*}\right)$ satisfying this following property:

$$
\exists\left(q^{*}, t^{*}\right) \in \mathrm{Q}\left(a^{*}, b^{*}\right) \text { such that } \forall i \in \mathcal{N},\left(a_{i}^{*}, b_{i}^{*}, q^{*}, t^{*}\right) \text { is solution of } P_{i}\left(a_{-i}^{*}, b_{-i}^{*}\right)
$$

It is interesting to observe that this spot market model, although it enters in the class of generalized Nash equilibrium problems, has a special structure in the sense that the problem $P_{i}\left(a_{-i}, b_{-i}\right)$ of the producer $i$ depends on the bids of the other producers only throughout the constraints, that is the shared ISO's problem. Indeed $P_{i}\left(a_{-i}, b_{-i}\right)$ is, formally, of the following form

$$
\begin{aligned}
\max _{x_{i}, y} & f_{i}\left(x_{i}, y\right) \\
\text { s.t. } & \left\{\begin{array}{l}
x_{i} \in X_{i} \\
y \in \arg \min _{K} g(x, \cdot)
\end{array}\right.
\end{aligned}
$$

3. Uniqueness in the ISO's problem. Since the constraint set of each producer's problem is essentially built around the constraint $(q, t) \in Q(a, b)$, it is important to know whether the solution set $Q(a, b)$ of the ISO's problem is a singleton. As shown by the following example, this is clearly not the case in general.

Example 3.0.1. Let us consider $\mathcal{N}=\{1 ; 2\}, \mathcal{L}=\{12 ; 21\}, D_{1}=D_{2}=1, \bar{Q}_{1}=$ $\bar{Q}_{2}=2, b_{1}=b_{2}=0, a_{1}=a_{2}=1$ and $L_{12}=L_{21}=0$. Then the solution set of the ISO's problem is described by

$$
\mathrm{Q}(a, b)=\left\{\begin{array}{l|l}
(q, t) \in[0 ; 2]^{2} \times\left(\mathbb{R}_{+}\right)^{2} & \begin{array}{l}
q_{1}+q_{2}=2 \\
t_{12}=\max \left(1-q_{2} ; 0\right) \\
t_{21}=\max \left(1-q_{1} ; 0\right)
\end{array}
\end{array}\right\}
$$

In subsection 3.1, we will first describe simple hypotheses ensuring uniqueness of the solution set of the ISO's problem. Then, in subsection 3.2, under the uniqueness
assumption, we will propose a reformulation of the producer's problem and derive qualitative properties of some elements of this reformulation.
3.1. Sufficient conditions for uniqueness. All along this subsection the bid vectors $a$ and $b$ (elements of $\mathbb{R}^{N}$ are assumed to be such that, for all $i \in \mathcal{N}$, $\underline{A}_{i} \leq a_{i} \leq \bar{A}_{i}$ and $\underline{B}_{i} \leq b_{i} \leq \bar{B}_{i}$. As previously stated the objective function of the ISO's problem $\operatorname{ISO}(a, b)$ is denoted by $g_{(a, b)}(q, t)=\sum_{i \in \mathcal{N}}\left(a_{i} q_{i}+b_{i} q_{i}^{2}\right)$ while $K$ stands for the constraint set, which doesn't depend on $a$ or $b$.

From the convexity of $K$ and the strict convexity of $g_{(a, b)}$ whenever the bid vector $b$ is positive (that is, $b_{i}>0$, for all $i \in \mathcal{N}$ ), we immediately deduce the following uniqueness result on variable $q$.
Proposition 1. We suppose that for all $i \in \mathcal{N}, b_{i}>0$. For all solutions $\left(q^{I}, t^{I}\right)$ and $\left(q^{I I}, t^{I I}\right)$ of the ISO's problem, we have $q^{I}=q^{I I}$.

The conclusion of the above proposition can be rephrased saying that all the solutions ( $q, t$ ) of the ISO's problem have the same production vector $q$.

The non uniqueness on the flow variable $t$ can be illustrated by the following simple example : $\mathcal{N}=\{1 ; 2 ; 3 ; 4\}, \mathcal{L}=\{12 ; 13 ; 24 ; 34\}, D_{1}=D_{2}=D_{3}=0, D_{4}=1$, $b_{1}=1=a_{1}, L_{12}=L_{24}=L_{13}=L_{34}=0$ and $a_{i}=b_{i}=4$ for $i=2,3,4$. Then there is a unique $q=(1,0,0,0)$ but $t^{I}=(.5, .5, .5, .5)$ and $t^{I I}=(.75, .25, .75, .25)$ correspond to different optimal solutions.

Nevertheless assuming that all the bid coefficients $b_{i}$ are positive could appear to be a bit too restrictive. In example 3.0.1, both $b_{1}$ and $b_{2}$ are assumed to be zero and there is a multiplicity of solutions. However, if $L_{12}=L_{21} \neq 0$, then there exists a unique solution to the ISO's problem which is given by $\left(q_{1} ; q_{2} ; t_{12} ; t_{21}\right)=(1 ; 1 ; 0 ; 0)$. Actually it is a general fact that the strict positivity of the thermal losses coefficients $L_{i j}$ will imply the desired uniqueness on the production vector $q$ and on the flow vector $t$. This will be stated in Proposition 2 below. Let us first prove the following useful lemma.
Lemma 3.1. Let us assume that for each producer $i \in \mathcal{N}$, one has $a_{i} \neq 0$ or $b_{i} \neq 0$. Then for any $(q, t) \in Q(a, b)$, solution of the ISO's problem $\operatorname{ISO}(a, b)$, one has, for any $i \in \mathcal{N}$,

$$
q_{i}-\sum_{k: i k \in \mathcal{L}}\left(t_{i k}+\frac{L_{i k}}{2} t_{i k}^{2}\right)+\sum_{k: k i \in \mathcal{L}}\left(t_{k i}-\frac{L_{k i}}{2} t_{k i}^{2}\right)=D_{i}
$$

Remark 1. i) The assumption that for any producer $i$, one has $a_{i} \neq 0$ or $b_{i} \neq 0$ simply means that there is no producer who proposes electricity for free.
ii) The equality in the above lemma expressed that Kirchhoff law is satisfied at any optimal solution of the ISO's problem.
Proof. Case 1): Let us first assume that $q_{i}>0$. The solution $(q, t)$ satisfies

$$
q_{i}-D_{i} \geq \sum_{k: i k \in \mathcal{L}}\left(t_{i k}+\frac{L_{i k}}{2} t_{i k}^{2}\right)-\sum_{k: k i \in \mathcal{L}}\left(t_{k i}-\frac{L_{k i}}{2} t_{k i}^{2}\right) .
$$

Let us suppose, for a contradiction, that the above inequality is strict. By defining

$$
q_{i}^{\prime}=\max \left\{\sum_{k: i k \in \mathcal{L}}\left(t_{i k}+\frac{L_{i k}}{2} t_{i k}^{2}\right)-\sum_{k: k i \in \mathcal{L}}\left(t_{k i}-\frac{L_{k i}}{2} t_{k i}^{2}\right)+D_{i} ; 0\right\}<q_{i}
$$

and $q^{\prime}=\left(q_{1}, \cdots, q_{i-1}, q_{i}^{\prime}, q_{i+1}, \cdots, q_{N}\right)$, we have $\left(q^{\prime}, t\right) \in K$ and, since $g_{(a, b)}$ is increasing on $\left[0,+\infty\left[, g_{(a, b)}\left(q^{\prime}, t\right)<g_{(a, b)}(q, t)\right.\right.$, which is a contradiction.

Case 2): Consider now that $q_{j}=0$ and

$$
\begin{equation*}
-D_{j}>\sum_{k: j k \in \mathcal{L}}\left(t_{j k}+\frac{L_{j k}}{2} t_{j k}^{2}\right)-\sum_{k: k j \in \mathcal{L}}\left(t_{k j}-\frac{L_{k j}}{2} t_{k j}^{2}\right) \tag{1}
\end{equation*}
$$

The latter means that the input flow into node $j$ is higher than output flow plus consumption at node $j$. We will show that, step by step, it is then possible to redistribute the flows in such a way that a contradiction will be obtained at node $i$ such that $q_{i}>0$.

So let a node $i \in \mathcal{N}$ be such that $q_{i}>0$ and there exists $\left\{i_{1}, \cdots, i_{p}\right\} \subset \mathcal{N}$ such that for any $k \in\{1, \cdots, p-1\}$

$$
i_{k} i_{k+1} \in \mathcal{L}, t_{i_{k} i_{k+1}}>0 \text { and } i_{1}=i, i_{p}=j
$$

To justify the existence of such a path $i_{1} i_{2}, i_{2} i_{3}, \ldots, i_{p-1} j$, let us observe that, since there is no production at node $j$, inequality (1) shows that the entering flow at node $j$, namely, $\sum_{k: k j \in \mathcal{L}}\left(t_{k j}-\frac{L_{k j}}{2} t_{k j}^{2}\right)$ is strictly positive because it is greater than $\sum_{k: j k \in \mathcal{L}}\left(t_{j k}+\frac{L_{j k}}{2} t_{j k}^{2}\right)+D_{j}$. Thus there is at least one strictly positive entering flow $l j$ at $j$. If $q_{l}>0$ then the path will be reduced to $l j$. Otherwise we can act recursively.

By continuity of the function $x \longrightarrow x-\frac{L_{i_{p-1} i_{p}}}{2} x^{2}$, there exists $\tilde{t}_{i_{p-1} i_{p}}<t_{i_{p-1} i_{p}}$ which satisfies

$$
\begin{align*}
-D_{j} \geq \sum_{k: j k \in \mathcal{L}}\left(t_{j k}+\frac{L_{j k}}{2} t_{j k}^{2}\right) & -\sum_{k: k j \in \mathcal{L} \backslash I_{p-1}}\left(t_{k j}-\frac{L_{k j}}{2} t_{k j}^{2}\right)  \tag{2}\\
& -\left(\tilde{t}_{i_{p-1} j}-\frac{L_{i_{p-1} j}}{2}\left(\tilde{t}_{i_{p-1} j}\right)^{2}\right)
\end{align*}
$$

where $I_{p-1}=\left\{i_{p-1} i_{p}\right\}$. On the other hand one can easily verify that

$$
\begin{aligned}
&-D_{i_{p-1}} \geq \sum_{k: i_{p-1}}\left(t_{i_{p-1} k}+\frac{L_{i_{p-1} k} k}{2} t_{i_{p-1} k}^{2}\right)+\left(\tilde{t}_{i_{p-1} j}+\frac{L_{i_{p-1}} j}{2}\left(\tilde{t}_{i_{p-1} j}\right)^{2}\right) \\
&-\sum_{k: k i_{p-1} \in \mathcal{L}}\left(t_{k i_{p-1}}-\frac{L_{k i_{p-1}}}{2} t_{k i_{p-1}}^{2}\right)
\end{aligned}
$$

proving, together with (2) that $\left(q, \tilde{t}^{p-1}\right) \in K$ where $\tilde{t}^{p-1}=\left(\left(t_{r s}\right)_{r s \in \mathcal{L} \backslash I_{p-1}}, \tilde{t}_{i_{p-1} i_{p}}\right)$.
By a finite reverse recurrence we will show that, for $k=p-1$ to $k=1$, one can find $\left.\tilde{t}^{k}=\left(\left(t_{r s}\right)_{r s \in \mathcal{L} \backslash I_{k}},\left(\tilde{t}_{r s}\right)_{r s \in I_{k}}\right)\right)$ with $I_{k}=\left\{i_{k} i_{k+1}, \ldots, i_{p-1} i_{p}\right\},\left(q, \tilde{t}^{k}\right) \in K$ and $\tilde{t}_{r s}<t_{r s}$, for all $r s \in I_{k}$.

So let us suppose the above recurrence property is true for $k$. Since $\left(q, \tilde{t}^{k}\right) \in K$ and $\tilde{t}_{i_{k} i_{k+1}}<t_{i_{k} i_{k+1}}$, we immediately have

$$
\begin{aligned}
q_{i_{k}}-D_{i_{k}} \geq & \sum_{l: i_{k} l \in \mathcal{L}}\left(t_{i_{k} l}+\frac{L_{i_{k} l}}{2} t_{i_{k} l}^{2}\right)-\sum_{k: l i_{k} \in \mathcal{L}}\left(t_{l i_{k}}-\frac{L_{l i_{k}}}{2} t_{l i_{k}}^{2}\right) \\
> & \sum_{l: i_{k} l \in \mathcal{L} \backslash I_{k}}\left(t_{i_{k} l}+\frac{L_{i_{k} l}}{2} t_{i_{k} l}^{2}\right)-\sum_{k: l i_{k} \in \mathcal{L}}\left(t_{l i_{k}}-\frac{L_{l i_{k}}}{2} t_{l i_{k}}^{2}\right) \\
& +\tilde{t}_{i_{k} i_{k+1}}+\frac{L_{i_{k} i_{k+1}}}{2}\left(\tilde{t}_{i_{k} i_{k+1}}\right)^{2}
\end{aligned}
$$

Therefore we can find $\tilde{t}_{i_{k-1} i_{k}}<t_{i_{k-1} i_{k}}$ such that

$$
\begin{aligned}
q_{i_{k}}-D_{i_{k}} \geq & \sum_{l: i_{k} l \in \mathcal{L} \backslash I_{k-1}}\left(t_{i_{k} l}+\frac{L_{i_{k} l}}{2} t_{i_{k} l}^{2}\right)-\sum_{l: l i_{k} \in \mathcal{L} \backslash I_{k-1}}\left(t_{l i_{k}}-\frac{L_{l i_{k}}}{2} t_{l i_{k}}^{2}\right) \\
& +\tilde{t}_{i_{k} i_{k+1}}+\frac{L_{i_{k} i_{k+1}}}{2}\left(\tilde{t}_{i_{k} i_{k+1}}\right)^{2}-\left(\tilde{t}_{i_{k-1} i_{k}}-\frac{L_{i_{k-1} i_{k}}}{2} \tilde{t}_{i_{k-1} i_{k}}^{2}\right) .
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
q_{i_{k-1}}-D_{i_{k-1}} \geq & \sum_{l: i_{k-1} l \in \mathcal{L} \backslash I_{k-1}}\left(t_{i_{k-1} l}+\frac{L_{i_{k-1} l} l}{2} t_{i_{k-1} l}^{2}\right)+t_{i_{k-1} i_{k}}^{\prime} \\
& +\frac{L_{i_{k-1} i_{k}}}{2}\left(t_{i_{k-1} i_{k}}^{\prime}\right)^{2}-\sum_{l: l i_{k-1} \in \mathcal{L}}\left(t_{l i_{k-1}}-\frac{L_{l i_{k-1}}}{2} t_{l i_{k-1}}^{2}\right)
\end{aligned}
$$

showing that $\left(q, \tilde{t}^{k-1}\right) \in K$ with $\left.\tilde{t}^{k-1}=\left(\left(t_{r s}\right)_{r s \in \mathcal{L} \backslash I_{k-1}},\left(\tilde{t}_{r s}\right)_{r s \in I_{k-1}}\right)\right)$
As a final conclusion of the recurrence (for $k=1$ ), the vector ( $q, \tilde{t}^{1}$ ) is in $K$ and

$$
q_{i}-D_{i}>\sum_{l: i l \in \mathcal{L} \backslash I_{1}}\left(t_{i l}+\frac{L_{i l}}{2} t_{i l}^{2}\right)-\sum_{l: l i \in \mathcal{L}}\left(t_{l i}-\frac{L_{l i}}{2} t_{l i}^{2}\right)+\tilde{t}_{i i_{2}}+\frac{L_{i i_{2}}}{2}\left(\tilde{t}_{i i_{2}}\right)^{2}
$$

with $q_{i}>0$. But we know, thanks to Case 1 ), that this is impossible. This completes the proof.

We are now in a position to prove a uniqueness result for the solution of the ISO's problem based on the strict positivity of the thermal losses coefficients.

Proposition 2. Let us assume that for all producers $i \in \mathcal{N}$, one has $a_{i} \neq 0$ or $b_{i} \neq 0$, and, for all lines $i j \in \mathcal{L}, L_{i j}>0$. Then the ISO's problem $\operatorname{ISO}(a, b)$ admits a unique solution $\left(q^{*}, t^{*}\right)$.

Proof. Let $\left(q^{I}, t^{I}\right)$ and $\left(q^{I I}, t^{I I}\right)$ be two solutions of the ISO's problem such that $t^{I} \neq t^{I I}$, and let $\left.\lambda \in\right] 0 ; 1\left[\right.$. By convexity of $K$ and $g_{(a, b)},\left(q^{\lambda}, t^{\lambda}\right):=\left(\lambda q^{I}+(1-\right.$ $\left.\lambda) q^{I I}, \lambda t^{I}+(1-\lambda) t^{I I}\right)$ is also a solution of the ISO's problem.

Since $t^{I} \neq t^{I I}$ there exists $i j \in \mathcal{L}$ such that $t_{i j}^{I} \neq t_{i j}^{I I}$ and therefore, by convexity of the functions $\varphi_{l p}: x \longrightarrow L_{l p} x^{2}$ and strict convexity of the function $\varphi_{i j}$, it follows that

$$
q_{j}^{\lambda}-\sum_{k: j k \in \mathcal{L}}\left(t_{j k}^{\lambda}+\frac{L_{j k}}{2}\left(t_{j k}^{\lambda}\right)^{2}\right)+\sum_{k: k j \in \mathcal{L}}\left(t_{k j}^{\lambda}-\frac{L_{k j}}{2}\left(t_{k j}^{\lambda}\right)^{2}\right)>D_{j}
$$

But the above inequality contradicts lemma 3.1. Thus we have $t^{I}=t^{I I}$ and, again by Lemma 3.1, for any $j, q_{j}^{I}=q_{j}^{I I}=D_{j}-\sum_{k: j k \in \mathcal{L}}\left(t_{j k}^{\lambda}+\frac{L_{j k}}{2}\left(t_{j k}^{\lambda}\right)^{2}\right)+$ $\sum_{k: k j \in \mathcal{L}}\left(t_{k j}^{\lambda}-\frac{L_{k j}}{2}\left(t_{k j}^{\lambda}\right)^{2}\right)$.

Whenever the quadratic bid function is supposed to be an approximation of some bids by blocks (a block being a couple (quantity-price)), assuming that $b_{i}>0$ corresponds to the fact that the stack of block bids is organized in increasing order and that the associated increasing step function can approximated by a quadratic form (see also remark after the definition of the producer's problem).
3.2. Reformulation and qualitative properties. Let us denote by $\mathcal{A}$ and $\mathcal{B}$ the subsets $\mathcal{A}=\prod_{i \in \mathcal{N}}\left[\underline{A}_{i} ; \bar{A}_{i}\right]$ and $\mathcal{B}=\prod_{i \in \mathcal{N}}\left[\underline{B}_{i} ; \bar{B}_{i}\right]$. All along this subsection, we will assume that, for all $(a, b) \in \mathcal{A} \times \mathcal{B}$, the ISO's problem $\operatorname{ISO}(a, b)$ admits a unique solution. This solution will be denoted by $(q(a, b), t(a, b))$. In the particular case when the bids of the producer act only on the linear term of the production cost function, that is whenever $b=B$ is fixed and known by all producers, we will use the notation $(q(a), t(a))$. In the literature (see e.g. [18]) this case is called "bid-on- $a$-only" market. Our aim, in this subsection, is then to investigate qualitative properties of the application $q:(a, b) \mapsto q(a, b)$.

Under the above uniqueness assumption, the producer's problem $P_{i}\left(a_{-i}, b_{-i}\right)$ can be reformulated in the following form:

$$
\begin{aligned}
\max _{a_{i}, b_{i}} & \left(a_{i}+2 b_{i} q_{i}(a, b)\right) q_{i}(a, b)-\left(A_{i} q_{i}(a, b)+B_{i} q_{i}(a, b)^{2}\right) \\
\text { s.t. } & \left\{\begin{array}{l}
\underline{A}_{i} \leq a_{i} \leq \bar{A}_{i} \\
\underline{B}_{i} \leq b_{i} \leq \bar{B}_{i}
\end{array}\right.
\end{aligned}
$$

Even if this reformulation is only an implicit form, it has the advantage of replacing the bilevel problem $P_{i}\left(a_{-i}, b_{-i}\right)$ by a classical mathematical programming problem. Additionally, as it will be shown in Section 4, it is possible, in some cases, to give an explicit formulation for $q(a, b)$ and therefore, the multi-leader-follower game which represents the spot electricity market becomes a classical Nash equilibrium problem.

Proposition 3. Let us suppose that $b=B$ is known by all the agents. Then, for all $a_{-i} \in \prod_{j \neq i}\left[\underline{A}_{i} ; \bar{A}_{i}\right]$, the application $q_{i}\left(\cdot, a_{-i}\right)$ is non increasing and Lipschitz continuous with modulus $\frac{1}{2 B_{i}}$ on $\left[\underline{A}_{i} ; \bar{A}_{i}\right]$.

Proof. It follows the same lines as in the proof of [18, Prop. 1]. Let $a$ and $a^{\prime}$ be two elements of $\mathcal{A}$. Since the constraint set $K$ and the objective function $g_{a}:(q, t) \mapsto$ $\sum_{i \in \mathcal{N}}\left(a_{i} q_{i}+b_{i} q_{i}^{2}\right)$ of the ISO's problem $I S O(a)$ are convex, we obtain, according to classical optimality conditions written at $q(a)$ and $q\left(a^{\prime}\right)$ elements of $K$

$$
\begin{aligned}
\left\langle 2 M q(a)+a, q\left(a^{\prime}\right)-q(a)\right\rangle & \geq 0 \\
\left\langle 2 M q\left(a^{\prime}\right)+a^{\prime}, q(a)-q\left(a^{\prime}\right)\right\rangle & \geq 0
\end{aligned}
$$

where $M$ stands for the diagonal matrix $M=\operatorname{diag}\left(B_{1}, \cdots, B_{N}\right)$. Summing those two inequalities, we obtain

$$
\left\langle 2 M\left(q(a)-q\left(a^{\prime}\right)\right)+a-a^{\prime}, q\left(a^{\prime}\right)-q(a)\right\rangle \geq 0
$$

and therefore

$$
\left\langle a^{\prime}-a, q\left(a^{\prime}\right)-q(a)\right\rangle \leq 2\left\langle M\left(q(a)-q\left(a^{\prime}\right)\right), q\left(a^{\prime}\right)-q(a)\right\rangle \leq 0 .
$$

The above inequality being satisfied for any $a^{\prime} \in \mathcal{A}$, we can consider, for any $i \in \mathcal{N}, a^{\prime}=\left(a_{1}, \ldots, a_{i-1}, a_{i}^{\prime}, a_{i+1}, \ldots, a_{n}\right)$ and thus it immediately follows that $\left(q_{i}\left(a_{i}^{\prime}, a_{-i}\right)-q_{i}\left(a_{i}, a_{-i}\right)\right)\left(a_{i}^{\prime}-a_{i}\right) \leq 0$ which proves that $q_{i}\left(\cdot, a_{-i}\right)$ is non increasing on $\left[\underline{A}_{i} ; \bar{A}_{i}\right]$.

Moreover, we have

$$
\left(a_{i}^{\prime}-a_{i}\right)\left(q_{i}\left(a_{i}^{\prime}, a_{-i}\right)-q_{i}\left(a_{i}, a_{-i}\right)\right) \leq-2 B_{i}\left(q_{i}\left(a_{i}^{\prime}, a_{-i}\right)-q_{i}\left(a_{i}, a_{-i}\right)\right)^{2} \leq 0
$$

showing that

$$
\left|q_{i}\left(a_{i}^{\prime}, a_{-i}\right)-q_{i}\left(a_{i}, a_{-i}\right)\right| \leq \frac{1}{2 B_{i}}\left|a_{i}^{\prime}-a_{i}\right| .
$$

Lemma 3.2. For any $(a, b) \in \mathcal{A} \times \mathcal{B}$ and any $i \in \mathcal{N}$, if the function $q_{i}\left(\cdot, a_{-i}, \cdot, b_{-i}\right)$ is continuously differentiable on a neighbourhood of $\left(a_{i}, b_{i}\right)$ then $q$ satisfies the following inviscid Burgers' equation

$$
\frac{\partial}{\partial b_{i}} q_{i}(a, b)-2 q_{i}(a, b) \frac{\partial}{\partial a_{i}} q_{i}(a, b)=0
$$

Proof. Let us consider the value function $v: \mathbb{R}^{N} \times \mathbb{R}^{N} \longrightarrow \mathbb{R}$ of the parametrized ISO's problem $\operatorname{ISO}(a, b)$. It is defined by

$$
v(a, b):=\min _{(q, t) \in K} \sum_{i \in \mathcal{N}}\left(a_{i} q_{i}+b_{i} q_{i}^{2}\right)=\sum_{i \in \mathcal{N}}\left(a_{i} q_{i}(a, b)+b_{i} q_{i}(a, b)^{2}\right) .
$$

By [6, Theorem 4.1], the value function $v$ is differentiable at $(a, b)$ with $\partial_{a_{i}} v(a, b)=$ $q_{i}(a, b)$ and $\partial_{b_{i}} v(a, b)=q_{i}^{2}(a, b)$. Then, according to the classical Schwarz's theorem, $\frac{\partial^{2}}{\partial a_{i} \partial b_{i}} v=\frac{\partial^{2}}{\partial b_{i} \partial a_{i}} v$, which yields to $\frac{\partial}{\partial b_{i}} q_{i}(a, b)=2 q_{i}(a, b) \frac{\partial}{\partial a_{i}} q_{i}(a, b)$.
4. The case of the New Zealand's market. In this section we shall concentrate on the case of the New Zealand market, or more generally, on the case of a two islands type market. We will give, in this case an explicit formula for the solutions of the ISO's problem and will prove the existence of a global equilibrium for the electricity spot market.

In what follows, one island is called the north island and the other is called the south island. This particular topography generates the following special structure of the network.


Figure 1. The two islands type network

Producers and consumers are distributed among the two islands. Two oriented lines are connecting the islands with the same thermal losses coefficient $L>0$. Taking the thermal losses is here of particular importance since, due to the distance between both islands and to the transmission technology (usually based on direct current), the losses of energy could be rather big. On the other hand, since the distance between the producers of the same island is quite small, it is reasonable to
assume that there are no thermal losses on the lines inside each island. This last assumption, together with the fact that any pair of producers of the same island are supposed to be connected, implies that the marginal price at all the nodes of one island are the same.

We denote by $\{1, \cdots, p-1\}$ the set of nodes of the north island, $\{p, \cdots, N\}$ the set of nodes of the south island. The lines connecting the north and the south islands are between node/producer 1 and node/producer $p$. The associated flows will be respectively denoted by $t_{1}$ on the line $1 p$ from the north to the south and by $t_{2}$ for the reverse one. We assume that the bids of the producers act only on the linear term of the production cost function, that is $b=B$ is fixed and known by all producers. Thus the two islands type ISO's problem reduces to the following convex problem $I S O_{T I}(a, B)$, denoted in the sequel by $I S O_{T I}(a)$,

$$
\begin{aligned}
\min _{q, t} & \sum_{i=1}^{N}\left(B_{i} q_{i}^{2}+a_{i} q_{i}\right) \\
\text { s.t. } & \left\{\begin{array}{l}
q_{i} \geq 0, \forall i \in \mathcal{N} \\
t_{k} \geq 0, \forall k \in\{1,2\} \\
\sum_{i=1}^{p-1} q_{i}-D_{N} \geq t_{1}-t_{2}+\frac{L}{2}\left(t_{1}^{2}+t_{2}^{2}\right) \\
\sum_{i=p}^{N} q_{i}-D_{S} \geq t_{2}-t_{1}+\frac{L}{2}\left(t_{1}^{2}+t_{2}^{2}\right)
\end{array}\right.
\end{aligned}
$$

with $D_{N}=\sum_{i=1}^{p-1} D_{i}$ and $D_{S}=\sum_{i=p}^{N} D_{i}$ representing respectively the total demand at the north and at the south. Let us observe that the capacity constraints (production and lines) are not taken into account in this formulation.

Market assumptions:
In the sequel we will use the following set $(\mathcal{H})$ of market assumptions for the ISO's problem of the two islands type electricity spot market
$\left(\mathcal{H}_{1}\right)$ linear bid of producer 1 and producer $p: B_{1}=B_{p}=0, a_{1}>0$ and $a_{p}>0$
$\left(\mathcal{H}_{2}\right)$ quadratic bid of the other producers : $B_{i}>0$, for any $i \neq 1, p$
$\left(\mathcal{H}_{3}\right)$ non trivial demand market: $D_{N}>0$ or $D_{S}>0$
Using Lemma 3.1, we can show that, at any solution of the ISO's problem, the Kirchhorff law type inequality constraints are active.

Lemma 4.1. Let us assume the market assumptions $\left(\mathcal{H}_{1-2}\right)$. Then for any $(q, t) \in$ $Q(a)$, solution of the ISO's problem $\operatorname{ISO} O_{T I}(a)$, one has,

$$
\sum_{i=1}^{p-1} q_{i}-D_{N}=t_{1}-t_{2}+\frac{L}{2}\left(t_{1}^{2}+t_{2}^{2}\right)
$$

and

$$
\sum_{i=p}^{N} q_{i}-D_{S}=t_{2}-t_{1}+\frac{L}{2}\left(t_{1}^{2}+t_{2}^{2}\right)
$$

Proof. Let us prove the equality for the north island. Since $a_{1}>0$ and $B_{i}>0$ for any $i=2, \ldots, p-1$, one has, according to Lemma 3.1 and to the fact that there are no thermal losses on the inside lines of the north island, $q_{1}-t_{1}-\frac{L}{2} t_{1}^{2}+$ $t_{2}-\frac{L}{2} t_{2}^{2}-\sum_{k=1: 1 k \in \mathcal{L}}^{p-1} t_{1 k}+\sum_{k=1: k 1 \in \mathcal{L}}^{p-1} t_{k 1}=D_{1}$ and, for any $i=2, \ldots, p-1$,
$q_{i}-\sum_{k=1: i k \in \mathcal{L}}^{p-1} t_{i k}+\sum_{k=1: k i \in \mathcal{L}}^{p-1} t_{k i}=D_{i}$. Now the expected equality follows by summing the above equalities, taking into account that $\sum_{i=1}^{p-1} \sum_{k=1: i k \in \mathcal{L}}^{p-1} t_{i k}=\sum_{i=1}^{p-1} \sum_{k=1: k i \in \mathcal{L}}^{p-1} t_{k i}$. The second equality is obtained by the same way.
Let us now provide a uniqueness result.
Proposition 4. Under the market assumptions $\left(\mathcal{H}_{1-2}\right)$ the two islands type ISO's problem $I S O_{T I}(a)$ admits a unique solution.
Remark 2. a) The bid functions of producers 1 and $p$ being assumed to be linear $\left(B_{1}=B_{p}=0\right)$, it is thus crucial not to have $a_{1}=0$ or $a_{p}=0$.
$b)$ None of the uniqueness results of subsection 3.1 can be applied to prove the above proposition. Indeed, since the thermal losses on the lines connecting producers of the same island are neglected, Proposition 2 cannot be used. On the other hand assumptions of Proposition 1 are not satisfied either since $B_{1}=B_{p}=0$.
$c)$ It is important to notice that the above uniqueness result only concerns the production vector $q$ and the flows $t_{1}$ and $t_{2}$. The thermal losses being neglected inside the islands, there is no uniqueness of the flows $t_{i j}$ inside the island (see the example following Proposition 1).

Proof. Let $\left(q^{I}, t^{I}\right)$ and ( $q^{I I}, t^{I I}$ ) be two different solutions of the ISO's problem $I S O_{T I}(a)$. For any $\left.\lambda \in\right] 0,1\left[,\left(q^{\lambda}, t^{\lambda}\right)=\lambda\left(q^{I}, t^{I}\right)+(1-\lambda)\left(q^{I I}, t^{I I}\right)\right.$ is also a solution of $I S O_{T I}(a)$.

The proof is divided into four steps.
Step 1: Let us first prove that $t^{I}=t^{I I}$. If $t_{1}^{I} \neq t_{1}^{I I}$ then, by strict convexity of function $(x, y) \longrightarrow \frac{L}{2}\left(x^{2}+y^{2}\right)$, we have

$$
\sum_{i=1}^{p-1} q_{i}^{\lambda}+D_{N}>t_{1}^{\lambda}-t_{2}^{\lambda}+\frac{L}{2}\left(\left(t_{1}^{\lambda}\right)^{2}+\left(t_{2}^{\lambda}\right)^{2}\right)
$$

providing a contradiction with Lemma 4.1. Now if $t_{1}^{I}=t_{1}^{I I}$ but $t_{2}^{I} \neq t_{2}^{I I}$ then, symmetrically,

$$
\sum_{i=p}^{N} q_{i}^{\lambda}-D_{S}>t_{2}^{\lambda}-t_{1}^{\lambda}+\frac{L}{2}\left(\left(t_{1}^{\lambda}\right)^{2}+\left(t_{2}^{\lambda}\right)^{2}\right)
$$

contradicting again Lemma 4.1.
Step 2: It is now clear that, for any $j \notin\{1, p\}, q_{j}^{I}=q_{j}^{I I}$. Indeed, if it exists $j \notin$ $\{1, p\}$ such that $q_{j}^{I} \neq q_{j}^{I I}$ then, by strict convexity of the function $x \mapsto B_{i} q_{i}^{2}+a_{i} q_{i}$, the objective function $g_{a}(q, t)=\sum_{k=1}^{N}\left(B_{k} q_{k}^{2}+a_{k} q_{k}\right)$ of the ISO's problem satisfies $g_{a}\left(q^{\lambda}\right)<g_{a}\left(q^{I}\right)$ which contradicts the fact that $\left(q^{I}, t^{I}\right)$ is a solution of the ISO's problem.

Step 3: $q_{1}^{I}=q_{1}^{I I}$. Indeed, according to Lemma 4.1 one has

$$
q_{1}^{I}+\sum_{i=2}^{p-1} q_{i}^{I}-D_{N}=t_{1}^{I}-t_{2}^{I}+\frac{L}{2}\left(\left(t_{1}^{I}\right)^{2}+\left(t_{2}^{I}\right)^{2}\right)
$$

and

$$
q_{1}^{I I}+\sum_{i=2}^{p-1} q_{i}^{I I}-D_{N}=t_{1}^{I I}-t_{2}^{I I}+\frac{L}{2}\left(\left(t_{1}^{I I}\right)^{2}+\left(t_{2}^{I I}\right)^{2}\right)
$$

and therefore, combining with Step 1 and Step $2, q_{1}^{I}=q_{1}^{I I}$.

Step 4: By the same arguments as in Step 3, it is now clear that $q_{p}^{I}=q_{p}^{I I}$. This completes the proof.

Let us observe that, since the thermal losses are neglected inside each island, all producers of the same island have the same marginal price, which will be denoted respectively by $\lambda_{N}$ for the north island and $\lambda_{S}$ for the south island.

For the following proposition, we introduce a notation: let two real numbers $x, y$, the notation $1_{[x \leq y]}$ denotes a real number which value is 0 if $x>y$ and 1 if $x \leq y$.

Proposition 5. Let $a \in \mathcal{A}$. Under the market assumptions $(\mathcal{H})$, at least one of the marginal prices $\lambda_{N}$ and $\lambda_{S}$ is strictly positive. Additionally the unique solution $(q(a), t(a))$ of the two islands ISO's problem ISO ${ }_{T I}(a)$ verifies

$$
t_{1}(a)=\frac{\lambda_{S}-\lambda_{N}}{L\left(\lambda_{N}+\lambda_{S}\right)} 1_{\left[\lambda_{N} \leq \lambda_{S}\right]} \quad \text { and } \quad t_{2}(a)=\frac{\lambda_{N}-\lambda_{S}}{L\left(\lambda_{N}+\lambda_{S}\right)} 1_{\left[\lambda_{S} \leq \lambda_{N}\right]}
$$

Moreover, if $q_{1}(a) \neq 0$ and $q_{p}(a) \neq 0$, we have

$$
t_{1}(a)=\frac{a_{p}-a_{1}}{L\left(a_{1}+a_{p}\right)} 1_{\left[a_{1} \leq a_{p}\right]} \quad \text { and } \quad t_{2}(a)=\frac{a_{1}-a_{p}}{L\left(a_{1}+a_{p}\right)} 1_{\left[a_{p} \leq a_{1}\right]}
$$

Remark 3. As an immediate consequence of the above formula, we have either $t_{1}(a)=0$ or $t_{2}(a)=0$. Actually $t_{2}(a) \neq 0$ if and only if the marginal price of the north island is strictly higher than the marginal price of the south island and, symetrically, $t_{1}(a) \neq 0$ if and only if the marginal price of the south island is strictly higher than the marginal price of the north island.

The rest of this section, including the proof of Proposition 5 is based on the following Karush-Kuhn-Tucker optimality sufficient conditions for convex programming problem $I S O_{T I}(a)$ :
For any producer $i \in\{1, \ldots, p-1\}$ of north island:

$$
\begin{equation*}
2 B_{i} q_{i}(a)+a_{i}-\mu_{i}-\lambda_{N}=0 \tag{i}
\end{equation*}
$$

For any producer $i \in\{p, \ldots, N\}$ of south island:

$$
\begin{equation*}
2 B_{i} q_{i}(a)+a_{i}-\mu_{i}-\lambda_{S}=0 \tag{ii}
\end{equation*}
$$

For all producers :

$$
\begin{array}{rcccc}
0 & \leq \mu_{i} \perp & q_{i}(a) & \geq 0 \\
0 & \leq \lambda_{N} \perp & g_{N}(q(a), t(a)) & \geq 0 \\
0 & \leq \lambda_{S} \perp & g_{S}(q(a), t(a)) & \geq 0 \tag{v}
\end{array}
$$

with $g_{N}(q, t)=t_{1}-t_{2}+\frac{L}{2}\left(t_{1}^{2}+t_{2}^{2}\right)-\sum_{i=1}^{p-1} q_{i}+D_{N}$ and $g_{S}(g, t)=t_{2}-t_{1}+\frac{L}{2}\left(t_{1}^{2}+\right.$ $\left.t_{2}^{2}\right)-\sum_{i=p}^{N} q_{i}+D_{S}$.
And concerning the lines $1 p$ and $p 1$

$$
\begin{gather*}
0 \leq \beta_{1} \perp t_{1}(a) \geq 0  \tag{vi}\\
\left.-\beta_{1}+\lambda_{N}\left(1+L t_{1}(a)\right)-\lambda_{S}\left(1-L t_{1}(a)\right)\right)=0  \tag{vii}\\
0 \leq \beta_{2} \perp t_{2}(a) \geq 0  \tag{viii}\\
\left.-\beta_{2}-\lambda_{N}\left(1-L t_{2}(a)\right)+\lambda_{S}\left(1+L t_{2}(a)\right)\right)=0 \tag{ix}
\end{gather*}
$$

Proof. of Proposition 5. Let us first assume, for a contradiction, that $\lambda_{N}=\lambda_{S}=0$. Since $B_{1}=0$, this leads, according to (i), to $\mu_{1}=a_{1}>0$ and thus, by (iii) to $q_{1}(a)=0$. Now for any $i=1, \ldots, p-1$, from $(i)$, we deduce that $\mu_{i}=2 B_{i} q_{i}(a)+a_{i}$ and thus, if $q_{i}(a)>0$ that $\mu_{i}>0$. But in this case, $(i i i)$ yields $q_{i}(a)=0$ which is impossible. As a conclusion, if $\lambda_{N}=\lambda_{S}=0$ then $q_{i}(a)=0$, for any $i=1, \ldots, N$, immediately implying that $t_{i j}=0$ for any $i j \in \mathcal{L}$ providing a contradiction between Lemma 4.1 and hypothesis $\left(\mathcal{H}_{3}\right)$.

From (vi) and (vii) we immediately have $\beta_{1}=\lambda_{N}\left(1+L t_{1}(a)\right)-\lambda_{S}\left(1-L t_{1}(a)\right)$ with $\beta_{1} t_{1}(a)=0$. If $\lambda_{N}>\lambda_{S}$ then $\beta_{1}$ is strictly positive and thus $t_{1}(a)=0$. Now if $\lambda_{N}<\lambda_{S}$ then $t_{1}(a) \neq 0$, since, otherwise, $t_{1}(a)=0$ would imply $\beta_{1}=\lambda_{N}-\lambda_{S} \geq 0$. So $\beta_{1}=0$ and consequently

$$
t_{1}(a)=\frac{\lambda_{S}-\lambda_{N}}{L\left(\lambda_{N}+\lambda_{S}\right)}
$$

Finally the last case to be considered is $\lambda_{N}=\lambda_{S}$. Then in this situation $\beta_{1}=$ $2 L \lambda_{N} t_{1}(a)$ and therefore $\beta_{1} t_{1}(a)=2 L \lambda_{N} t_{1}^{2}(a)$. Since $\lambda_{N}>0$ we have $t_{1}(a)=0$.

Similar calculus can be done for $t_{2}(a)$. Now if $q_{1}(a) \neq 0$ and $q_{p}(a) \neq 0,(i i i)$ yields $\mu_{1}=\mu_{p}=0$ and thus, according to (i) and (ii), $\lambda_{N}=a_{1}$ and $\lambda_{S}=a_{p}$.

Lemma 4.2. Let $a \in \mathcal{A}$ be such that $q_{1}(a) \neq 0$ and $q_{p}(a) \neq 0$. If the two islands type market satisfies $(\mathcal{H})$ then

$$
\begin{aligned}
q_{1}(a)+q_{p}(a) & =D_{N}+D_{S}+\frac{1}{L}\left[\left(\frac{a_{p}-a_{1}}{a_{1}+a_{p}}\right)^{2} 1_{\left[a_{1} \leq a_{p}\right]}+\left(\frac{a_{1}-a_{p}}{a_{1}+a_{p}}\right)^{2} 1_{\left[a_{p} \leq a_{1}\right]}\right] \\
& +\sum_{i \text { at north } i \neq 1} \frac{a_{i}-a_{1}}{2 B_{i}} 1_{\left[a_{i} \leq a_{1}\right]}+\sum_{i \text { at south } i \neq p} \frac{a_{i}-a_{p}}{2 B_{i}} 1_{\left[a_{i} \leq a_{p}\right]}
\end{aligned}
$$

Proof. According the KKT's conditions (i), (ii) and (iii), we have, for all $i \in$ $\mathcal{N} \backslash\{1 ; p\}, \mu_{i}=0$ and

$$
\left\{\begin{array}{l}
2 B_{i} q_{i}(a)+a_{i}-\lambda_{N}+\mu_{i}=0 \text { if } i=2, \ldots, p-1 \\
2 B_{i} q_{i}(a)+a_{i}-\lambda_{S}+\mu_{i}=0 \text { if } i=p+1, \ldots, N
\end{array}\right.
$$

which immediately leads to

$$
\begin{aligned}
& q_{i}(a)+\frac{a_{i}-\lambda_{N}}{2 B_{i}} 1_{\left[q_{i}(a) \neq 0\right]}=0 \text { if } i=2, \ldots, p-1 \\
& q_{i}(a)+\frac{a_{i}-\lambda_{S}}{2 B_{i}} 1_{\left[q_{i}(a) \neq 0\right]}=0 \quad \text { if } i=p+1, \ldots, N .
\end{aligned}
$$

Since $q_{1}(a) \neq 0$ and $q_{p}(a) \neq 0$ we have, according to $(i)$ and $(i i), \lambda_{N}=a_{1}$ and $\lambda_{S}=a_{p}$. Adding the above equalities (over $i \in\{2, \cdots, p-1\} \cup\{p+1, \cdots, N\}$ ), we obtain

$$
\sum_{i \neq 1, p} q_{i}(a)+\sum_{i \text { at }} \sum_{\text {north }} \frac{a_{i}-a_{1}}{2 B_{i}} 1_{\left[q_{i} \neq 0\right]}+\sum_{i \text { at south }} \frac{a_{i}-a_{p}}{2 B_{i}} 1_{\left[q_{i} \neq 0\right]}=0
$$

By lemma 4.1, we have $\sum_{i \neq 1, p} q_{i}(a)=L\left(t_{1}(a)^{2}+t_{2}(a)^{2}\right)+D_{N}+D_{S}-q_{1}(a)-q_{p}(a)$, which proves that:

$$
\begin{aligned}
q_{1}(a)+q_{p}(a)= & L\left(t_{1}(a)^{2}+t_{2}(a)^{2}\right)+D_{N}+D_{S} \\
& +\sum_{i \text { at north }} \frac{a_{i}-a_{1}}{2 B_{i}} 1_{\left[q_{i} \neq 0\right]}+\sum_{i \text { at south } i \neq p} \frac{a_{i}-a_{p}}{2 B_{i}} 1_{\left[q_{i} \neq 0\right]}
\end{aligned}
$$

Let us now observe that, for the north island producers $i=2, \cdots, p-1, a_{i}<a_{1}$ if and only if $q_{i}(a) \neq 0$. Indeed if $q_{i}(a)>0$ then, since $B_{i}>0$, from (ii) we have $-a_{i}+a_{1}=-a_{i}+\lambda_{N}=2 B_{i} q_{i}(a)>0$. On the other hand, if $a_{i}<a_{1}$, according to (ii), we obtain $a_{i}-\lambda_{N}=\mu_{i}-2 B_{i} q_{i}(a)$ and thus, if $q_{i}(a)=0$, a contradiction occurs since $a_{i}-a_{1}=a_{i}-\lambda_{N}=\mu_{i} \geq 0$.

Finally the desired expression of $q_{1}(a)+q_{p}(a)$ follows by combining the above calculus with Proposition 5.

In the above Lemma 4.2 the parameters and variables of the north and south islands play symmetric roles. It is thus natural, knowing the uniqueness result on $I S O_{T I}$ and taking into account that only one of the quantities $t_{1}(a)$ and $t_{2}(a)$ are non zero, to deduce the expression of the unique solution $(q(a), t(a))$ of $I S O_{T I}(a)$.

Proposition 6. Let $a \in \mathbb{R}_{+}^{N}$ be such that the market assumptions $(\mathcal{H})$ holds and $q_{1}(a) \neq 0$ and $q_{p}(a) \neq 0$. Then

$$
\begin{aligned}
& q_{1}(a)=\frac{\left(a_{p}-a_{1}\right)\left(a_{1}+3 a_{p}\right)}{2 L\left(a_{1}+a_{p}\right)^{2}}+\sum_{i \text { at north }} \frac{a_{i}-a_{1}}{2 B_{i}} 1_{\left[a_{i} \leq a_{1}\right]}+D_{N} \\
& q_{p}(a)=\frac{\left(a_{1}-a_{p}\right)\left(3 a_{1}+a_{p}\right)}{2 L\left(a_{1}+a_{p}\right)^{2}}+\sum_{i \text { at south }} \frac{a_{i}-a_{p}}{2 B_{i}} 1_{\left[a_{i} \leq a_{p}\right]}+D_{S}
\end{aligned}
$$

Additionally,
For $i=2, \ldots, p-1, q_{i}(a)=\frac{a_{1}-a_{i}}{2 B_{i}} 1_{\left[a_{i} \leq a_{1}\right]}$ and for $i=p+1, \ldots, N-1$, $q_{i}(a)=\frac{a_{p}-a_{i}}{2 B_{i}} 1_{\left[a_{i} \leq a_{1}\right]}$.

Based on the above formula, we prove quasiconcavity of the objective functions $p_{i}\left(a_{i}, a_{-i}\right)=\left(a_{i}-A_{i}\right) q_{i}(a)$ of the producer's problems.

Proposition 7. Let $a \in \mathbb{R}_{+}^{N}$ be such that the market assumptions $(\mathcal{H})$ hold and $q_{1}(a) \neq 0$ and $q_{p}(a) \neq 0$. Then for any $a \in \mathcal{A}$ one has
a) for any $i \in \mathcal{N} \backslash\{1, p\}$, the objective function $p_{i}\left(\cdot, a_{-i}\right)$ is quasiconcave over $\mathcal{A}_{i}=\left[\underline{A}_{i}, \bar{A}_{i}\right] ;$
b) the objective function $p_{1}\left(\cdot, a_{-1}\right)$ is quasiconcave over $\left[A_{1}, 3 A_{1}\right] \cap\left[\underline{A}_{1}, \bar{A}_{1}\right]$;
c) the objective function $p_{p}\left(\cdot, a_{-p}\right)$ is quasiconcave over $\left[A_{p}, 3 A_{p}\right] \cap\left[\underline{A}_{p}, \bar{A}_{p}\right]$;
where $A_{1}$ and $A_{p}$ are the linear coefficients of the real cost functions of producers 1 and $p$ respectively.

Proof. Let us first consider $i \in\{2, \ldots, p-1\}$. According to Proposition 6, $q_{i}\left(\cdot, a_{-i}\right)$ is piecewise linear and the unique point of non differentiability is $a_{1}$. For all $a_{i}<a_{1}$, we have $q_{i}\left(a_{i}, a_{-i}\right)=\frac{a_{1}-a_{i}}{2 B_{i}}$, thus $\partial_{i} p_{i}\left(a_{i}, a_{-i}\right)=-\frac{1}{2 B_{i}}<0$. On the other hand for all $a_{i}>a_{1}, q_{i}\left(a_{i}, a_{-i}\right)=0$ and therefore $p_{i}\left(a_{i}, a_{-i}\right)=0$. It is thus clear that $p_{i}\left(\cdot, a_{-i}\right)$ is quasiconcave on $\mathcal{A}_{i}$. The same arguments can be used to argue on the quasiconcavity of $p_{i}\left(\cdot, a_{-i}\right)$ for $i \in\{p+1, \ldots, N\}$.

Let us now prove the quasiconcavity of the function $p_{1}\left(\cdot, a_{-1}\right)$ on $\left[A_{1}, 3 A_{1}\right] \cap$ $\left[\underline{A}_{1}, \bar{A}_{1}\right]$. Let us denote by $I_{1}=\left\{i \in\{2, \ldots, p-1\}: a_{i}<a_{1}\right\}$. On any segment $] a_{i}, a_{j}\left[, i, j \in I_{1}\right.$, the function is two times differentiable and concave. Indeed,

$$
\begin{aligned}
& \frac{\partial}{\partial a_{1}} p_{1}(a)=q_{1}(a)+\left(a_{1}-A_{1}\right) \frac{\partial}{\partial a_{1}} q_{1}(a) \text { and } \\
& \frac{\partial^{2}}{\partial a_{1}^{2}} p_{1}\left(a_{1}, a_{-1}\right)=2 \frac{\partial}{\partial a_{1}} q_{1}\left(a_{1}, a_{-1}\right)+\left(a_{1}-A_{1}\right) \frac{\partial^{2}}{\partial a_{1}^{2}} q_{1}\left(a_{1}, a_{-1}\right) \\
& \leq-\frac{8 a_{p}^{2}}{L\left(a_{1}+a_{2}\right)^{3}}+\left(a_{1}-A_{1}\right) \frac{12 a_{p}^{2}}{L\left(a_{1}+a_{p}\right)^{4}} \\
&=\frac{4 a_{p}^{2}\left(a_{1}-2 a_{p}-3 A_{1}\right)}{2 L\left(a_{1}+a_{p}\right)^{4}} \\
& \leq 0
\end{aligned}
$$

It is now sufficient to prove that at any $i \in I_{1}$, one has

$$
\lim _{w \nearrow a_{i}} \frac{\partial}{\partial a_{1}} p_{1}\left(w, a_{-1}\right)<0 \Longrightarrow \lim _{w \searrow a_{i}} \frac{\partial}{\partial a_{1}} p_{1}\left(w, a_{-1}\right) \leq 0 .
$$

If $a_{1}=A_{1}$ the above implication is an immediate consequence of the continuity of the application $q_{1}\left(\cdot, a_{-1}\right)$. Otherwise, it is equivalent to

$$
\begin{equation*}
\lim _{w \nearrow a_{i}} \frac{\partial}{\partial a_{1}} q_{1}\left(w, a_{-1}\right)<-\frac{q_{1}(a)}{a_{1}-A_{1}} \Longrightarrow \lim _{w \nearrow a_{i}} \frac{\partial}{\partial a_{1}} q_{1}\left(w, a_{-1}\right)<-\frac{q_{1}(a)}{a_{1}-A_{1}} \tag{3}
\end{equation*}
$$

According to the expression of $q_{1}$ obtained in Proposition 6, there exists $r>0$ such that for all $u \in] a_{i}-r ; a_{i}[, v \in] a_{i} ; a_{i}+r[$,

$$
\frac{\partial}{\partial a_{1}} q_{1}\left(v ; a_{-1}\right)=\frac{\partial}{\partial a_{1}} q_{1}\left(u ; a_{-1}\right)-\sum_{j \in \tilde{I}_{1}^{i}} \frac{1}{2 B_{j}}
$$

where $\tilde{I}_{1}^{i}=\left\{j \in I_{1}: a_{j}=a_{i}\right\}$. Therefore (3) is fulfilled. Thus $\partial_{1} q_{1}\left(a_{1}^{-} ; a_{-1}\right) \geq$ $\partial_{1} q_{1}\left(a_{1}^{+} ; a_{-1}\right)$, which proves the quasiconcavity of $p_{1}\left(\cdot, a_{-1}\right)$ on $\mathcal{A}_{1}$.

We are now in a position to prove, by classical arguments (see e.g., [20, Theorem 4.1]), the existence of an equilibrium for the two islands type electricity market.

Corollary 1. Assume that $0<A_{1}<\underline{A}_{1} \leq \bar{A}_{1}<3 A_{1}, 0<A_{p}<\underline{A}_{p} \leq \bar{A}_{p}<3 A_{p}$ and that the market assumptions $(\mathcal{H})$ are satisfied. If, for any $a \in \mathcal{A}$, the ISO's dispatch on producers 1 and $p$ is nonzero then the two islands type electricity spot market admits at least a solution.

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