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# Gap Functions for Quasivariational Inequalities and Generalized Nash Equilibrium Problems

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**Abstract** The gap function (or merit function) is a classic tool for reformulating a Stampacchia variational inequality as an optimization problem. In this paper, we adapt this technique for quasivariational inequalities, that is, variational inequalities in which the constraint set depends on the current point. Following Fukushima (J. Ind. Manag. Optim. 3:165–171, 2007), an axiomatic approach is proposed. Error bounds for quasivariational inequalities are provided and an application to generalized Nash equilibrium problems is also considered.

**Keywords** Gap function · Merit function · Set-valued map · Quasivariational inequality · Nash equilibrium

## 1 Introduction

Variational inequalities provide a perfect setting to express a large number of problems coming from different areas, such as mechanics, optimality conditions, equilibrium problems in economics. Motivated by the richness of the algorithmic machinery of optimization, an important effort has been made to reformulate variational inequalities in terms of an optimization problem (see [2–7] and references therein). This reformulation is done thanks to the concept of *gap function* (or merit function).

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Nevertheless, for some specific applications (like, e.g., generalized Nash equilibrium problems), it is necessary to consider variational inequalities, where the constraint set depends on the current point, that is, quasivariational inequalities $QVI(T,S)$ where $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $S:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are two set-valued maps.

Recently, in Dietrich [8] and in Fukushima [1], some advances have been made to define gap functions for quasivariational inequalities. In [1], the operator $T$ defining the quasivariational inequality is supposed to be single-valued and differentiable, while in [8] the set-valued case is considered but the function, which plays the role of the gap function, depends on both variables $x$ and $x^* \in T(x)$. Our aim in this paper is to define a (single variable) gap function for general set-valued quasivariational inequalities.

The paper is organized as follows: following [1], in Sect. 2 we define gap function through an axiomatic approach. Section 3 is devoted to error bounds and also provides uniqueness results. Finally, in Sect. 4, using a new way to reformulate generalized Nash equilibrium problems (GNEP, for short) as quasivariational inequalities, we define a gap function for GNEP.

## 2 Axiomatic Gap Functions for QVI

Let us first introduce some notations, which will be used in the sequel. Throughout the paper, $\mathbb{R}^n$ is equipped with the Euclidian norm $\|\cdot\|$ associated with the scalar product $\langle\cdot,\cdot\rangle$.

Given a set-valued map $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and a subset $K$ of $\mathbb{R}^n$, the finite-dimensional variational inequality $VI(T,K)$ is

$$\begin{aligned}&\text{Find } \bar{x}\in K \text{ such that there exists } \bar{x}^*\in T(\bar{x})\\ &\text{with } \langle \bar{x}^*, y-\bar{x}\rangle \geq 0, \quad \forall y\in K\end{aligned}$$

while a quasivariational inequalities $QVI(T,S)$ is

$$\begin{aligned}&\text{Find } \bar{x}\in S(\bar{x}) \text{ such that there exists } \bar{x}^*\in T(\bar{x})\\ &\text{with } \langle \bar{x}^*, y-\bar{x}\rangle \geq 0, \quad \forall y\in S(\bar{x})\end{aligned}$$

where $S:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map.

For $x\in\mathbb{R}^n$ and $\rho>0$, we denote by $B(x,\rho)$ and $\overline{B}(x,\rho)$, respectively, the open and the closed ball with centre $x$ and radius $\rho$, while for $x,x'\in\mathbb{R}^n$, $[x,x']$ stands for the closed segment $\{tx+(1-t)x' : t\in[0,1]\}$. The segments $]x,x'[$, $]x,x']$, $[x,x'[$ are defined analogously. The topological closure and the interior of a set $A\subset\mathbb{R}^n$ will be denoted, respectively, by $\mathrm{cl}A$ (or $\bar{A}$) and $\mathrm{int}A$, while $\mathbb{R}^*_+A$ stands for the conic hull of $A$ that is $\mathbb{R}^*_+A := \{\lambda a : \lambda>0, a\in A\}$. Given a set-valued operator $T$ defined on $\mathbb{R}^n$, its domain and its graph will be denoted, respectively, by $\mathrm{dom}\,T$ and $\mathrm{Gr}T$, while $FP(T)$ is the set of fixed points of $T$ and, for any subset $U$ of $\mathbb{R}^n$, $T(U)$ stands for $T(U):=\bigcup_{u\in U} T(u)$. For any bivariate function $\varphi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ convex in the second variable, $\partial_2\varphi(x,y)$ denotes the Fenchel subdifferential of the function $\varphi(x,\cdot)$, that is, $\partial_2\varphi(x,y) := \{y^*\in\mathbb{R}^n : \langle y^*, z-y\rangle \leq \varphi(x,z)-\varphi(x,y)\ \forall z\in\mathbb{R}^n\}$.

Finally, let us recall that a set-valued operator $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be

– *lower semicontinuous* at $x_0$ iff, for any sequence $(x_n)_n$ of $\mathbb{R}^n$ converging to $x_0$ and any element $x_0^*$ of $T(x_0)$, there exists a sequence $(x_n^*)_n$ of $\mathbb{R}^n$ converging to $x_0^*$ such that, for any $n$, $x_n^* \in T(x_n)$;
– *upper semicontinuous* at $x_0 \in \operatorname{dom} T$ iff, for any neighbourhood $V$ of $T(x_0)$, there exists a neighbourhood $U$ of $x_0$ such that $T(U) \subset V$;
– *closed graph* at $x_0 \in \operatorname{dom} T$ iff, for any sequence $((x_n, x_n^*))_n \in \mathrm{Gr}(T)^{\mathbb{N}}$ converging to $(x_0, x_0^*)$, one has $(x_0, x_0^*) \in \mathrm{Gr}(T)$. Another usual terminology is that $\mathrm{Gr}(T)$ is closed at $x_0$;
– *$\mu$-strongly monotone* ($\mu \geq 0$) on a subset $K$ iff, for all $x, y \in K$ and all $x^* \in T(x)$, $y^* \in T(y)$,

$$\langle y^* - x^*, y - x\rangle \geq \mu\|y - x\|^2.$$

Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be two set-valued operators with $\operatorname{dom} S \subset \operatorname{dom} T$. We will say that the couple $(T, S)$ satisfies assumption $(H)$ iff there exists a function $\varphi : \operatorname{dom} S \times \mathbb{R}^n \to \mathbb{R}$ verifying the following properties:

(B1) for all $x \in \operatorname{dom} S$, $\varphi(x, .)$ is strictly convex on $\operatorname{dom} S$;
(B2) for all $x \in \operatorname{dom} S$, $\partial_2\varphi(x, x) = T(x)$.

It is important to note that properties (B1)–(B2) do not imply any monotonicity property of $T$. Indeed, $T$ is not the subdifferential map of a given convex function but rather, at each point $x$, the value $T(x)$ is the subdifferential at $x$ of a function depending on $x$, namely $\varphi(x, \cdot)$. A characterization of (B1)–(B2) will be given in the forthcoming Corollary 2.1.

Let us also remark that, if $(H)$ holds true, then, $T(x)$ being the subdifferential of a finite convex function for any $x \in \operatorname{dom} S$, the map $T$ is convex and compact valued on $\operatorname{dom} S$. The compactness of the values of $T$ could be considered a somewhat strong assumption. Actually, the following proposition shows that, using a classical "compactification process" of $T$, it is not the case.

**Proposition 2.1** *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be two set-valued maps. If $T$ is convex and nonempty valued, then the operator $\tilde{T} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, defined by*

$$\tilde{T}(x) := \begin{cases} \{0\}, & \text{if } 0 \in T(x), \\ \mathit{conv}\{\frac{x^*}{\|x^*\|} \mid x^* \in T(x)\}, & \text{if } 0 \notin T(x), \end{cases}$$

*is convex and compact valued on* $\operatorname{dom} S$ *and both quasivariational inequalities $QVI(T, S)$ and $QVI(\tilde{T}, S)$ have the same solutions.*

*Proof* Let $x \in \operatorname{dom} S$. Without loss of generality we can assume that $0 \notin T(x)$ since otherwise $\tilde{T}(x) = \{0\}$ and $x$ is a trivial solution of $QVI(T, S)$ and $QVI(\tilde{T}, S)$. It is now sufficient to prove that $\mathbb{R}_+^* T(x) = \mathbb{R}_+^* \tilde{T}(x)$, since in this case $x$ is solution of $QVI(T, S)$ if and only if $x$ is solution of $QVI(\tilde{T}, S)$. Clearly, one has $\mathbb{R}_+^* T(x) \subset \mathbb{R}_+^* \tilde{T}(x)$. Let us show the reverse inclusion and assume that $x^* \in \mathbb{R}_+^* \tilde{T}(x)$. Then there exist real numbers $\lambda > 0$, $\mu_1, \ldots, \mu_n \in \mathbb{R}_+$ and elements $x_1^*, \ldots, x_n^* \in T(x)$

such that $\sum_1^n \mu_k = 1$ and $x^* = \lambda \sum_1^n \mu_k \frac{x_k^*}{\|x_k^*\|}$. Set

$$M := \left(\lambda \sum \frac{\mu_k}{\|x_k^*\|}\right)^{-1} > 0 \quad \text{and} \quad \lambda_k = \frac{M\lambda\mu_k}{\|x_k^*\|}, \quad k = 1, \dots, n.$$

Since $\lambda_k \geq 0$, $\sum_1^n \lambda_k = 1$ and $T(x)$ is convex, we have $Mx^* = \sum \lambda_k x_k^* \in T(x)$ and therefore $x^* \in \mathbb{R}_+^* T(x)$. □

Let us now give a simple example of a possible definition for the bivariate function $\varphi$ such that a given couple $(T, S)$ satisfies hypothesis $(H)$.

**Proposition 2.2** *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be two set-valued maps such that* dom $S \subset$ dom $T$. *If $T$ is convex and compact valued, then, for any $\beta > 0$, the application $\varphi_\beta$ :* dom $S \times \mathbb{R}^n \to \mathbb{R}$ *defined, for any $(x, y) \in$* dom $S \times \mathbb{R}^n$ *by*

$$\varphi_\beta(x, y) := \sup_{x^* \in T(x)} \langle x^*, y - x\rangle + \beta\|y - x\|^2,$$

*satisfies conditions* (B1)–(B2). *If, moreover, $S$ has closed and convex values, then, for all $x \in$* dom $S$, *the function $\varphi_\beta(x, .)$ admits a (unique) minimizer over $S(x)$.*

*Proof* For any $x \in$ dom $S$, let us consider the function $g : \mathbb{R}^n \to \mathbb{R}$ given by

$$g(y) = \sup_{z^* \in T(x)} \langle z^*, y - x\rangle.$$

We clearly have $\varphi_\beta(x, y) = g(y) + \beta\|y - x\|^2$ and $\partial_2\varphi_\beta(x, x) = \partial g(x)$. On the other hand, since $T$ is convex compact valued, $\partial g(x) = T(x)$ and thus property (B2) is satisfied.

For any $x \in$ dom $S$, one immediately observes that $\varphi_\beta(x, \cdot)$ is strongly convex over $\mathbb{R}^n$ (and thus (B1) holds) and thus, if $S(x)$ is closed and convex, $\varphi_\beta(x, \cdot)$ admits a unique minimizer over $S(x)$. □

Combining the above proposition with the fact that any map satisfying (B1)–(B2) is convex and compact valued (see remarks following the definitions of (B1)–(B2)), we obtain the following characterization of hypothesis $(H)$.

**Corollary 2.1** *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be two set-valued maps such that domS $\subset$ domT. The couple $(T, S)$ satisfies $(H)$ if and only if $T$ has convex and compact values on* dom $S$.

One can wonder whether, given a couple $(T, S)$, the bivariate function $\varphi_\beta$ described in Proposition 2.2 corresponds to the unique way of constructing a bivariate function $\varphi$ such that $(H)$ be satisfied. The following example shows that it is not the case.

*Example 2.1* Let $T:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by $T(x)=\bar{B}(g(x),r(x))$, where $g:\mathbb{R}^n\to\mathbb{R}^n$ and $r:\mathbb{R}^n\to\mathbb{R}_+^*$. An easy calculus shows that, for any $\beta>0$, the associated function $\varphi_\beta$ described by Proposition 2.2 is given by

$$\varphi_\beta(x,y)=\big\langle g(x),y-x\big\rangle+r(x)\|y-x\|+\beta\|y-x\|^2.$$

But another possible choice for $\varphi$ for which $(T,S)$ satisfies $(H)$ is

$$\forall(x,y)\in\big(\mathbb{R}^n\big)^2,\quad \varphi(x,y)=\frac{r(x)}{h'(0)}h\big(\|y-x\|\big)+\big\langle g(x),y-x\big\rangle,$$

where $h$ is any function from $\mathbb{R}$ to $\mathbb{R}$ satisfying the following properties: $h$ is strictly convex, is increasing on $\mathbb{R}_+$ and admits a derivative at 0 and $h'(0)\neq 0$.

Indeed, let us show that (B1) and (B2) are satisfied for $\varphi$. Firstly, one can easily prove the strict convexity of the function $h(\|\cdot\|)$, using the strict convexity and increasingness of the function $h$ and taking into account that, for any $z_1\neq z_2\in\mathbb{R}^n$, $\|z_1\|\neq\|z_2\|$ if $z_1\in\mathbb{R}_+z_2$ or $z_2\in\mathbb{R}_+z_1$ and $\|\lambda z_1+(1-\lambda)z_2\|<\lambda\|z_1\|+(1-\lambda)\|z_2\|$ otherwise. Hence, by translation and since $\frac{r(x)}{h'(0)}>0$, the function $\varphi(x,\cdot)$ is strictly convex. On the other hand, for any $x\in\operatorname{dom}S$,

$$\partial_2\varphi(x,x)=\frac{r(x)}{h'(0)}\partial\big(h\big(\|\cdot-x\|\big)\big)(x)+g(x).$$

Since the function $h$ is convex and $h'(0)\neq 0$, we have

$$\partial\big(h\big(\|\cdot-x\|\big)\big)(x)=\partial\big(h\circ\|.\|\big)(0)=h'(0)\partial\big(\|.\|\big)(0)=h'(0)\bar{B}(0,1),$$

thus proving that $\partial_2\varphi(x,x)=T(x)$.

**Proposition 2.3** *Let $T:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ and $S:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ be a couple of set-valued operators satisfying $(H)$ (with a function $\varphi$) and such that* $\operatorname{dom}S\subset\operatorname{dom}T$. *If the application $z\to\varphi(z,z)$ is lower semicontinuous at $x\in\operatorname{int}(\operatorname{dom}S)$ and, for all $y\in\mathbb{R}^n$, the functions $\varphi(\cdot,y)$ are upper semicontinuous at $x$, then* $\operatorname{Gr}(T)$ *is closed at $x$.*

One can immediately deduce from the above proposition that there is no hope to construct a continuous function $\varphi$ satisfying (B1) and (B2) for the couple $(T,S)$ if $T$ is not closed graph on $\operatorname{dom}S$.

*Proof* Let $((x_n,x_n^*))_n\in\operatorname{Gr}(T)^{\mathbb{N}}$ converges to $(x,x^*)$. Let $y\in\mathbb{R}^n$. Since $x$ is an element of $\operatorname{int}(\operatorname{dom}S)$, for $n$ large enough, one has $T(x_n)=\partial_2\varphi(x_n,x_n)$ and thus $\langle x_n^*,y-x_n\rangle\le\varphi(x_n,y)-\varphi(x_n,x_n)$. Therefore, by upper semicontinuity of $\varphi(\cdot,y)$ at $x$ and lower semicontinuity of $z\to\varphi(z,z)$, it follows that

$$\begin{aligned}\langle x^*,y-x\rangle&\le\limsup_{n\to+\infty}\big(\varphi(x_n,y)-\varphi(x_n,x_n)\big)\\&\le\limsup_{n\to+\infty}\varphi(x_n,y)-\liminf_{n\to+\infty}\varphi(x_n,x_n)\\&\le\varphi(x,y)-\varphi(x,x).\end{aligned}$$

Since the above inequality holds for any $y \in \mathbb{R}$, $x^* \in \partial_2\varphi(x,x) = T(x)$ and the proof is complete. □

As observed in [1], for different purposes—in particular for algorithms—it could be interesting to replace the constraint set-valued map $S$ with a more tractable local approximation of $S$. For example if, for any point $x$, $S(x)$ is determined by a set of differentiable inequalities:

$$S(x) = \big\{y \in \mathbb{R}^n : g_i(x, y) \leq 0,\ i = 1, \dots, n\big\},$$

then one can consider a polyhedral approximation map $\Gamma$ (see [1] for more details) described by the first order development of the functions $g_i$, that is,

$$\Gamma(x) = \big\{y \in \mathbb{R}^n : g_i(x, x) + \big\langle \nabla g_i(x, x), y - x\big\rangle \leq 0,\ i = 1, \cdot, n\big\}.$$

Let us recall that for any convex set $C$ of $\mathbb{R}^n$ and any point $x$ of $C$ the tangent cone to $C$ at $x$, denoted by $T_C(x)$ is defined by

$$T_C(x) := \mathbb{R}_+\big(C - \{x\}\big) = \big\{\lambda(c - x) : \lambda \geq 0,\ c \in C\big\}.$$

**Theorem 2.1** *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be two set-valued maps such that $S$ is convex valued. Let $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be such that*

(C1) *for any $x \in \mathbb{R}^n$, $S(x) \subset \Gamma(x)$*;
(C2) *the maps $S$ and $\Gamma$ have the same fixed points*;
(C3) *for any $x \in \mathbb{R}^n$, $\Gamma(x)$ is convex and nonempty*;
(C4) *for any $x \in \mathbb{R}^n$, $T_{S(x)}(x) = T_{\Gamma(x)}(x)$.*

*Then $QVI(T, S)$ and $QVI(T, \Gamma)$ have the same solutions.*

*Proof* From (C1) and (C2), clearly any solution of $QVI(T, \Gamma)$ is also a solution of $QVI(T, S)$. So let us now suppose that a point $x$ is a solution of $QVI(T, S)$. Thus there exists $x^* \in T(x)$ such that $\langle x^*, y - x\rangle \geq 0$, for any $y \in S(x)$. But taking into account (C4) and the definition of the tangent cone to a convex set, it follows that

$$\mathbb{R}_+\big(S(x) - \{x\}\big) = T_{S(x)}(x) = T_{\Gamma(x)}(x) = \mathbb{R}_+\big(\Gamma(x) - \{x\}\big)$$

and thus $x$ is also a solution of $QVI(T, \Gamma)$. □

Based on the previous results we are now in a position to define the concept of axiomatic gap function for quasivariational inequality, that is, a gap function defined using the axiomatic bivariate function $\varphi$ of hypothesis $(H)$.

If $\operatorname{dom} S \subset \operatorname{int}(\operatorname{dom} T)$ then, for any $x \in \operatorname{dom} S$, one immediately has $\operatorname{dom} S \subset \operatorname{int}(\operatorname{dom}\varphi(x, \cdot))$, and therefore, by condition (B2), $x$ solves $QVI(T, S)$ if and only if $x \in S(x)$ and

$$\exists x^* \in T(x) = \partial_2\varphi(x, \cdot)(x) \quad \text{such that } \langle x^*, y - x\rangle \geq 0, \quad \forall y \in S(x).$$

If $S$ is convex valued, by (B1) and classical nonsmooth optimality conditions, then $x$ is a solution of the latter quasivariational inequality if and only if $x$ is the global minimizer of $\varphi(x, .)$ over $S(x)$. This is summarized in the following lemma.

**Lemma 2.1** *Let* $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *and* $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be a couple of set-valued maps satisfying* $(H)$ (*with a function* $\varphi$) *and such that* $S$ *is convex valued and* $\operatorname{dom} S \subset \operatorname{int}(\operatorname{dom} T)$. *Then for any* $x \in \mathbb{R}^n$,

$$x \text{ solves } QVI(T, S) \quad \Leftrightarrow \quad x \text{ is the global minimizer of } \varphi(x, .) \text{ over } S(x).$$

Let us consider the function $f_\varphi$ defined on dom $S$ by

$$f_\varphi(x) = -\inf_{y \in S(x)} \varphi(x, y). \tag{1}$$

**Theorem 2.2** *Let* $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *and* $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be a couple of set-valued maps satisfying* $(H)$ (*with a function* $\varphi$) *and such that* $S$ *is convex valued and* $\operatorname{dom} S \subset \operatorname{int}(\operatorname{dom} T)$.

*If the function* $\varphi$ *satisfies* $\varphi(x, x) = 0$, *for any* $x \in \operatorname{dom} S$, *then* $f_\varphi$ *is a gap function for* $QVI(T, S)$ *on* $FP(S)$, *that is*, $f_\varphi(x) \geq 0$, *for any* $x \in FP(S)$ *and* $f_\varphi(x) = 0$ *if and only if* $x$ *solves* $QVI(T, S)$.

*Proof* For any fixed point $x$ of $S$, one clearly has $f_\varphi(x) \geq -\varphi(x, x) = 0$. On the other hand, according to Lemma 2.1, $x$ is a solution of $QVI(T, S)$ if and only if $x$ is the global minimizer of $\varphi(x, .)$ over $S(x)$ which is equivalent to $f_\varphi(x) = -\inf_{y \in S(x)} \varphi(x, y) = -\varphi(x, x) = 0$. □

## 3 Error Bounds for QVI

In order to establish error bounds for quasivariational inequalities, we will now concentrate on the particular bivariate function $\varphi_\beta$ described in Proposition 2.2 and therefore, the considered gap function will be, for any $\beta > 0$, the function $f_\beta : \operatorname{dom} S \to \mathbb{R}$ given by

$$f_\beta(x) := -\inf_{y \in S(x)} \varphi_\beta(x, y)$$

where $\varphi_\beta : \operatorname{dom} S \times \mathbb{R}^n \to \mathbb{R}$ defined, for any $(x, y) \in \operatorname{dom} S \times \mathbb{R}^n$ by

$$\varphi_\beta(x, y) := \sup_{x^* \in T(x)} \langle x^*, y - x \rangle + \beta \|y - x\|^2.$$

A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be *fixed point symmetric* if it satisfies the following property:

$$\forall x \in FP(S), \quad \big(S(x), x\big) \subset \operatorname{Gr}(S).$$

This property can be reformulated in the following geometrical form:

$$\forall x \in FP(S), \quad \forall y \in S(x) \text{ one has } x \in S(y).$$

**Proposition 3.1** *Let* $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *and* $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be a couple of set-valued maps with* $\operatorname{dom} S \subset \operatorname{dom} T$ *and such that* $T$ *is* $\mu$*-strongly monotone* ($\mu > 0$) *with nonempty* *compact values and S is fixed point symmetric. Let $\bar{x} \in \mathbb{R}^n$ be a solution of QVI$(T, S)$. Then $\bar{x}$ is the unique solution of QVI$(S, T)$ on $S(\bar{x})$ and, for any $\beta < \mu$ and any $x \in S(\bar{x})$, one has*

$$\|x - \bar{x}\| \le \frac{1}{\sqrt{\mu - \beta}} \sqrt{f_\beta(x)}.$$

*Remark 3.1* Let us observe that the above error bound results, as the forthcoming ones of this section, still holds if the $\mu$-strong monotonicity assumption on $T$ is weakened to the following $\mu$-strong pseudomonotonicity hypothesis: for all $x, y \in \mathbb{R}^n$,

$$\exists x^* \in T(x) : \langle x^*, y - x\rangle > 0 \quad \Rightarrow \quad \forall y^* \in T(y) : \langle y^*, y - x\rangle \ge \mu \|y - x\|^2.$$

*Proof* From the fixed point symmetric hypothesis on $S$, clearly $\bar{x}$ is an element of $S(x)$ and thus by considering $y = \bar{x}$ in the definition of the gap function $f_\beta$, it holds that

$$f_\beta(x) \ge \inf_{x^* \in T(x)} \langle x^*, x - \bar{x}\rangle - \beta \|x - \bar{x}\|^2.$$

Therefore by compactness of $T(x)$, there exists $x^*_{\bar{x}} \in T(x)$ such that

$$f_\beta(x) \ge \langle x^*_{\bar{x}}, x - \bar{x}\rangle - \beta \|x - \bar{x}\|^2.$$

On the other hand $x \in S(\bar{x})$ and thus there exists $\bar{x}^* \in T(\bar{x})$ such that $\langle \bar{x}^*, x - \bar{x}\rangle \ge 0$ which immediately implies, by $\mu$-strong monotonicity of $T$, that $\langle x^*_{\bar{x}}, x - \bar{x}\rangle \ge \mu \|x - \bar{x}\|^2$. Finally

$$f_\beta(x) \ge (\mu - \beta) \|x - \bar{x}\|^2.$$

Since $\mu > \beta$, for all $x \ne \bar{x}$, one has $f_\beta(x) > 0$, which proves that $\bar{x}$ is the unique solution of $QVI(S, T)$ on $S(\bar{x})$. Moreover, we have $\|x - \bar{x}\| \le \sqrt{\frac{f_\beta(x)}{\mu - \beta}}$. □

A simple, but useful, example for which the set-valued map $S$ is fixed point symmetric corresponds to the case where the problem $QVI(T, S)$ is actually a variational inequality $VI(T, K)$. In this case, $S$ being constant, the fixed point symmetric property is clearly satisfied and therefore we immediately obtain, as a particular case of Proposition 3.1, an error bound for the variational inequality $VI(T, K)$.

**Corollary 3.1** *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a $\mu$-strongly monotone $(\mu > 0)$ set-valued map with nonempty compact values and $K$ be a nonempty convex subset of $\mathbb{R}^n$. Let $\bar{x} \in \mathbb{R}^n$ be the (unique) solution of VI$(T, K)$. Then, for any $\beta < \mu$ and for any $x \in K$, one has*

$$\|x - \bar{x}\| \le \frac{1}{\sqrt{\mu - \beta}} \sqrt{f_\beta(x)}.$$

*where the gap function $f_\beta$ is given by*

$$f_\beta(x) = -\inf_{y \in K} \sup_{x^* \in T(x)} \langle x^*, y - x\rangle + \beta \|y - x\|^2.$$

Let us now give an error bound for the quasivariational inequality $QVI(T,S)$ in which the "fixed point symmetric" property on $S$ is replaced by an Hölder-type hypothesis. Let us recall that a set-valued map $S$ is said to be *locally $\alpha$-Hölder* ($\alpha > 0$) at a point $x \in \operatorname{dom} S$ if there exist $r > 0$ and $L > 0$ such that for all $x \in B(\bar{x}, r)$

$$S(\overline{x}) \cap B(\overline{x}, r) \subset S(x) + \bar{B}\big(0, L\|x - \bar{x}\|^{\alpha}\big).$$

**Proposition 3.2** *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a couple of set-valued maps with* $\operatorname{dom} S \subset \operatorname{dom} T$ *and such that $T$ is $\mu$-strongly monotone* ($\mu > 0$). *Assume that $T(K)$ be compact for any compact subset $K$ and that $S$ is locally $\alpha$-Hölder with $\alpha > 2$ at a solution $\bar{x} \in \operatorname{dom} T$. Let the real $\eta \in ]0, \min\{r, 1\}[$ be such that $\rho_\eta = \mu - LM\eta^{\alpha-2} - \beta(1 - 2L - L^2) > 0$, where $M = \sup\{\|x^*\|\ ,\ x^* \in T(\bar{B}(\bar{x}, 1))\}$.*

*Then $\bar{x}$ is the unique solution of $QVI(S,T)$ on $B(\bar{x}, \eta) \cap S(\bar{x})$, and for all $x \in B(\bar{x}, \eta) \cap S(\overline{x})$,*

$$\|x - \bar{x}\| \le \sqrt{\frac{f_\beta(x)}{\rho_\eta}}.$$

*Proof* Let $x \in B(\bar{x}, \eta) \cap S(\overline{x})$, and $z \in S(x)$ be such that $\|\bar{x} - z\| \le L\|\bar{x} - x\|^{\alpha}$. From the definitions, we immediately have

$$\begin{aligned} f_\beta(x) &= -\inf_{y \in S(x)} \sup_{x^* \in T(x)} \big\{\langle x^*, y - x\rangle + \beta\|y - x\|^2\big\} \\ &= \sup_{y \in S(x)} \inf_{x^* \in T(x)} \big\{\langle x^*, x - y\rangle - \beta\|y - x\|^2\big\} \\ &\ge \inf_{x^* \in T(x)} \big\{\langle x^*, x - z\rangle - \beta\|z - x\|^2\big\}. \end{aligned}$$

Since $T(x)$ is compact, there exists $x^* \in T(x)$ such that

$$\begin{aligned} f_\beta(x) &\ge \langle x^*, x - z\rangle - \beta\|z - x\|^2 \\ &\ge \langle x^*, x - \bar{x}\rangle + \langle x^*, \bar{x} - z\rangle - \beta\big(\|z - \bar{x}\| + \|\bar{x} - x\|\big)^2. \end{aligned} \tag{2}$$

By assumption, $\bar{x}$ solves $QVI(T,S)$ and $x \in S(\bar{x})$. Consequently there exists $\bar{x}^* \in T(\bar{x})$ such that $\langle \bar{x}^*, x - \bar{x}\rangle \ge 0$ and therefore, by $\mu$-strong monotonicity of $T$, $\langle x^*, x - \bar{x}\rangle \ge \mu\|x - \bar{x}\|^2$. Taking into account that $\|\bar{x} - x\| < 1$, inequality (2) becomes

$$\begin{aligned} f_\beta(x) &\ge \mu\|x - \bar{x}\|^2 - M\|\bar{x} - z\| - \beta\big(\|z - \bar{x}\| + \|\bar{x} - x\|\big)^2 \\ &\ge \mu\|x - \bar{x}\|^2 - ML\|\bar{x} - x\|^{\alpha} - \beta\big(L\|x - \bar{x}\|^{\alpha} + \|\bar{x} - x\|\big)^2 \\ &= \mu\|x - \bar{x}\|^2 - ML\|\bar{x} - x\|^{\alpha} \\ &\quad - \beta\big(L^2\|x - \bar{x}\|^{2\alpha} + 2L\|x - \bar{x}\|^{\alpha+1} + \|x - \bar{x}\|^2\big) \\ &\ge \big(\mu - ML\|\bar{x} - x\|^{\alpha-2} - \beta - 2\beta L - \beta L^2\big)\|x - \bar{x}\|^2 \\ &\ge \big(\mu - ML\eta^{\alpha-2} - \beta - 2\beta L - \beta L^2\big)\|x - \bar{x}\|^2 \\ &= \rho_\eta\|x - \bar{x}\|^2. \end{aligned}$$

Then for all $x \in B(\bar{x}, \eta) \cap S(\bar{x})$, if $x \neq \bar{x}$ we have $f_\beta(x) > 0$ because $\rho_\eta > 0$, thus proving that $\bar{x}$ is the unique solution of $QVI(S, T)$ over $B(\bar{x}, \eta) \cap S(\bar{x})$. Moreover, for all $x \in B(\bar{x}, \eta) \cap S(\bar{x})$, one has $\|x - \bar{x}\| \leq \sqrt{\frac{f_\beta(x)}{\rho_\eta}}$. □

*Remark 3.2* It is clear from the proof that the locally $\alpha$-Hölder hypothesis in the above proposition can be weakened by simply assuming that there exist three reals $\alpha > 2$, $L > 0$ and $r \in ]0, 1[$ such that for any $y \in B(\bar{x}, r)$, $\mathrm{dist}(\bar{x}, S(y)) \leq L\|y - \bar{x}\|^\alpha$.

In the quasivariational inequality $QVI(T, S)$, the set-valued map $S$ describes the constraints. It is thus important to consider the classic case, where the constraint set $S(x)$ is described by inequalities. In the following proposition, we provide, in this case, sufficient conditions for $S$ to be locally Hölder. To simplify we consider the case of one inequality, but the general case could be deduced by similar arguments.

**Proposition 3.3** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and $g : \mathbb{R}^n \to \mathbb{R}$ be $\alpha$-Hölder continuous on $\mathbb{R}^n$. Let us suppose that the constraint map $S$ be defined, for any $x \in \mathrm{dom}\, S$, by*

$$S(x) = \{y \in \mathbb{R}^n \mid f(y) \leq g(x)\}.$$

*Let $\overline{x} \in S(\bar{x})$ be such that $\nabla f(\bar{x}) \neq 0$. Then the constraint map $S$ is locally $\alpha$-Hölder at $\bar{x}$.*

*Proof* Let $M > 0$ be the $\alpha$-Hölder constant of $g$, that is, for any $(x_1, x_2)$ in $(\mathbb{R}^n)^2$, $|g(x_2) - g(x_1)| \leq M\|x_2 - x_1\|^\alpha$. Since $\nabla f(\bar{x}) \neq 0$, there exists $\eta > 0$ such that for all $x \in \bar{B}(\bar{x}, \eta)$, $\nabla f(x) \neq 0$. By continuity on $\bar{B}(\bar{x}, \eta)^2$ of the function $(x_1, x_2) \to \langle \nabla f(x_1), \frac{\nabla f(x_2)}{\|\nabla f(x_2)\|} \rangle$, and since this function is strictly positive at $(\bar{x}, \bar{x})$, there exist $m > 0$ and $\varepsilon \in ]0, \eta[$ such that

$$\left\langle \nabla f(x_1), \frac{\nabla f(x_2)}{\|\nabla f(x_2)\|} \right\rangle \geq m, \quad \forall (x_1, x_2) \in \bar{B}(\bar{x}, \varepsilon)^2. \tag{3}$$

Let $r > 0$ be chosen such that $r + r^\alpha M/m < \varepsilon$ and let $x$ be an element of $B(\bar{x}, r)$. If $g(\bar{x}) \leq g(x)$, then $S(\bar{x}) \subset S(x) \subset S(x) + \bar{B}(0, L\|x - \bar{x}\|^\alpha)$. Now let us suppose that $g(\bar{x}) > g(x)$. For any $y \in S(\bar{x}) \cap B(\bar{x}, r)$, we set $z = y - \frac{M}{m}\|x - \bar{x}\|^\alpha \frac{\nabla f(y)}{\|\nabla f(y)\|}$. According to the classical mean value theorem, there exists $\xi \in ]y, z[$ such that

$$f(y) - f(z) = \frac{M\|x - \bar{x}\|^\alpha}{m} \left\langle \nabla f(\xi), \frac{\nabla f(y)}{\|\nabla f(y)\|} \right\rangle. \tag{4}$$

Since $\|z - \bar{x}\| \leq \|z - y\| + \|y - \bar{x}\| \leq \frac{M}{m}\|x - \bar{x}\|^\alpha + r \leq \frac{M}{m} r^\alpha + r < \varepsilon$, the couple $(\xi, y)$ is an element of $B(\bar{x}, \varepsilon)^2$ and, combining (3) and (4), one has $f(y) - f(z) \geq M\|x - \bar{x}\|^\alpha$. On the other hand, since $f(y) \leq g(\bar{x})$ and $g(\bar{x}) - g(x) \leq M\|x - \bar{x}\|^\alpha$, we have $f(z) \leq g(\bar{x}) - M\|x - \bar{x}\|^\alpha \leq g(x)$, thus showing that $z$ is an element of $S(x)$. Consequently

$$\mathrm{dist}(y, S(x)) \leq \|y - z\| = \frac{M}{m}\|x - \bar{x}\|^\alpha$$

and $y \in S(x) + \bar{B}(0, L\|x - \bar{x}\|^\alpha)$ with $L = M/m$. □

## 4 Application to Generalized Nash Equilibrium Problem

The generalized Nash equilibrium problem (GNEP) is a Nash game in which each player's strategy depends on the other players' strategies. More precisely, assume that there be $p$ players and each player $\nu$ controls variables $x^\nu \in \mathbb{R}^{n_\nu}$. In fact, $x^\nu$ is a strategy of the player $\nu$. Let us denote by $x$ the vector of strategies

$$x = \left(x^1, \dots, x^p\right) \quad \text{and} \quad n = n_1 + n_2 + \dots + n_p.$$

Denote by $x^{-\nu}$ the vector formed by all players' decision variables except the player $\nu$. So we can also write $x = (x^\nu, x^{-\nu})$, which is a shortcut (already used in many papers on the subject; see e.g. [4, 9]) to denote the vector $x = (x^1, \dots, x^{\nu-1}, x^\nu, x^{\nu+1}, \dots, x^p)$. The strategy of the player $\nu$ belongs to the set $X_\nu(x^{-\nu})$, which obviously depends on the decision variables of the other players. The aim of the player $\nu$, given the strategy $x^{-\nu}$, is to choose a strategy $x^\nu$ such that $x^\nu$ solves the following optimization problem:

$$(P_\nu) \qquad \min_{x^\nu} \theta_\nu\left(x^\nu, x^{-\nu}\right), \quad \text{subject to} \quad x^\nu \in X_\nu\left(x^{-\nu}\right),$$

where $\theta_\nu(x^\nu, x^{-\nu})$ denotes the loss the player $\nu$ suffers when the rival players have chosen the strategy $x^{-\nu}$. The GNEP (Generalized Nash Equilibrium Problem) is to find $\bar{x} \in \mathbb{R}^n$ such that for all $\nu \in \{1, \dots, p\}$:

$$\bar{x}^\nu \in \arg \min_{X_\nu(\bar{x}^{-\nu})} \theta_\nu\left(\cdot, \bar{x}^{-\nu}\right).$$

A large number of applications in economics and engineering can be modelled as generalized Nash Equilibrium problem (see [4, 9] and references therein). In order to study the GNEP and to have efficient computational processes some reformulation of GNEP have been given. Connections with quasivariational inequalities have been investigated (see Bensoussan [10], Harker [11]) in the particular case whenever the loss functions $\theta_\nu$ are convex and differentiable with respect to the $\nu$th variable. Recently, a reformulation of the GNEP as a variational inequality has been obtained [12], in the quasiconvex case, under the assumption of the Rosen' law, that is, the existence of a nonempty subset $X$ of $\mathbb{R}^n$ such that, for any $\nu$, the set $X_\nu(x^{-\nu})$ is given as

$$X_\nu\left(x^{-\nu}\right) := \left\{x^\nu \in \mathbb{R}^{n_\nu} : \left(x^\nu, x^{-\nu}\right) \in X\right\}.$$

Our aim in this section is, imitating the so-called *normal approach* of [12], firstly to obtain a reformulation of the GNEP as a quasivariational inequality and, secondly, based on this reformulation, to deduce from the results of Sect. 2 a gap function for the generalized Nash equilibrium problem.

Now let us define the set-valued map $X : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by the product $X(x) = X_1(x^{-1}) \times \dots \times X_N(x^{-N})$, where $X_\nu : \mathbb{R}^{n-\nu} \rightrightarrows 2^{\mathbb{R}^{n_\nu}}$.

Let us recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *quasiconvex* iff for any $\lambda \in \mathbb{R}$, its sublevel $S_\lambda$ is convex, where

$$S_\lambda := \left\{y \in \mathbb{R}^n \mid f(y) \le \lambda\right\}$$

and $f$ is said to be *semistrictly quasiconvex* iff $f$ is quasiconvex and, for any $x$, $y \in \mathbb{R}^n$ with $f(x) \neq f(y)$ and $\lambda \in ]0, 1[$, one has

$$f\big(\lambda x + (1-\lambda)y\big) < \max\big\{f(x), f(y)\big\}.$$

Roughly speaking, a semistrictly quasiconvex function is a quasiconvex function, which does not admit "flat part", except possibly $\arg\min_{\mathbb{R}^n} f$. Finally the strict sublevel set of $f$ will be denoted by $S_\lambda^< = \{y \in \mathbb{R}^n : f(y) < \lambda\}$.

Before defining precisely the normal approach, we need to recall the definition of *adjusted sublevel set* of a given function $f : \mathbb{R}^n \to \mathbb{R}$:

$$S_f^a(x) := S_{f(x)} \cap \overline{B\,(S_{f(x)}^<, \rho_x)},$$

where $\rho_x = \operatorname{dist}(x, S_{f(x)}^<)$, if $x \notin \arg\min_{\mathbb{R}^n} f$ and $S_f^a(x) = S_{f(x)}$ otherwise. Here $B(S_{f(x)}^<, \rho_x) = S_{f(x)}^< + \rho_x B(0,1)$ denotes the $\rho_x$-neighbourhood of the set $S_{f(x)}^<$ where $B(0,1)$ is the unit ball in $\mathbb{R}^n$.

Now, given a quasiconvex function $f : \mathbb{R}^n \to \mathbb{R}$, the *normal operator* associated with $f$ is the set-valued map $N_f^a : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ which is given as

$$N_f^a(x) := \big\{v \in \mathbb{R}^n : \langle v, y - x\rangle \le 0,\ \forall y \in S_f^a(x)\big\},$$

that is, the set $N_f^a(x)$ is the polar cone to the translated adjusted sublevel set $S_f^a(x) - \{x\}$. It is important to observe that, in the case of a semistrictly quasiconvex function, $N_f^a(x)$ is simply the polar cone to the classical sublevel set $S_{f(x)}$ or to the strict sublevel set $S_{f(x)}^<$, that is, for any $x \notin \arg\min_{\mathbb{R}^n} f$,

$$N_f^a(x) = (S_{f(x)} - x)^\circ = \big(S_{f(x)}^< - x\big)^\circ.$$

We will denote, for any $\nu$ and any $x \in \mathbb{R}^n$, by $S_\nu(x)$ and $A_\nu(x^{-\nu})$ the subsets of $\mathbb{R}^{n_\nu}$

$$S_\nu(x) := S_{\theta_\nu(\cdot, x^{-\nu})}^a\big(x^\nu\big) \quad \text{and} \quad A_\nu\big(x^{-\nu}\big) := \arg\min_{\mathbb{R}^{n_\nu}} \theta_\nu\big(\cdot, x^{-\nu}\big).$$

In order to reformulate the GNEP as a quasivariational inequality problem we define the following set-valued map $N_\theta^a : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ which is described, for any $x = (x^1, \dots, x^p) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p}$, by

$$N_\theta^a(x) := F_1(x) \times \cdots \times F_p(x),$$

where

$$F_\nu(x) := \begin{cases} \overline{B}_\nu(0,1), & \text{if } x^\nu \in A_\nu(x^{-\nu}),\\ \operatorname{conv}(N_{\theta_\nu}^a(x^\nu) \cap S_\nu(0,1)), & \text{otherwise}, \end{cases}$$

with $\overline{B}_\nu(0,1)$ and $S_\nu(0,1)$ denoting the closed unit ball and the unit sphere of $\mathbb{R}^{n_\nu}$ and $N_{\theta_\nu}^a(x^\nu)$ standing for the normal operator of the quasiconvex function $\theta_\nu(\cdot, x^{-\nu})$ at $x^\nu$, that is,

$$N_{\theta_\nu}^a(x^\nu) := \big\{v^\nu \in \mathbb{R}^{n_\nu} : \big\langle v^\nu, u^\nu - x^\nu\big\rangle \le 0,\ \forall u^\nu \in S_\nu(x)\big\}.$$

Adapting, respectively, the proof of Lemma 3.1, Theorem 3.1 and Theorem 4.1 of Aussel-Dutta [12], one can obtain the following links between the generalized Nash equilibrium problem and the quasivariational inequality $QVI(N^a_\theta, X)$.

**Lemma 4.1** *Let $\nu \in \{1, \dots, p\}$. If the function $\theta_\nu$ is continuous quasiconvex with respect to the $\nu$th variable, then*

$$0 \in F_\nu(\bar{x}) \quad \Longleftrightarrow \quad \bar{x}^\nu \in A_\nu\big(\bar{x}^{-\nu}\big).$$

**Theorem 4.1** *Let us assume that, for any $\nu$, the function $\theta_\nu$ be continuous and quasiconvex with respect to the $\nu$th variable. Then every solution of $QVI(N^a_\theta, X)$ is a solution of the GNEP.*

**Theorem 4.2** *Let us suppose that the map $X$ be convex valued and, for any $\nu$, the loss function $\theta_\nu$ is continuous and semistrictly quasiconvex with respect to the $\nu$th variable.*

*Then $\bar{x}$ is a solution of the GNEP if and only if $\bar{x}$ is a solution of the variational inequality $QVI(N^a_\theta, X)$.*

As a conclusion of this section we construct a gap function for the generalized Nash equilibrium problem. Using the above notations for the GNEP, let us define, for any $\beta > 0$, the function $f_\beta : \operatorname{dom} X \to \mathbb{R}$ by

$$f_\beta(x) = -\inf_{y \in X(x)} \varphi_\beta(x, y)$$

where $\varphi_\beta : \operatorname{dom} X \times \mathbb{R}^n \to \mathbb{R}$ defined, for any $(x, y) \in \operatorname{dom} X \times \mathbb{R}^n$, by

$$\varphi_\beta(x, y) = \sup_{x^* \in N^a_\theta(x)} \langle x^*, y - x \rangle + \beta \|y - x\|^2.$$

**Proposition 4.1** *Let us suppose that the map $X$ be convex valued and, for any $\nu$, the loss function $\theta_\nu$ is continuous on $\mathbb{R}^n$ and semistrictly quasiconvex with respect to the $\nu$th variable.*

*Then the application $f_\beta$ is a gap function for the generalized Nash equilibrium problem, that is, for any $x \in FP(X)$, $f_\beta(x) \geq 0$ and $f_\beta(x) = 0$ if and only if $x$ is a solution of GNEP.*

*Proof* For all $x \in \mathbb{R}^n$, $N^a_\theta(x)$ is compact and convex. Thus by Proposition 2.2, $\varphi_\beta$ satisfies (B1)–(B2), that is the couple $(N^a_\theta, X)$ satisfies hypothesis $(H)$. Thus according to Theorem 2.2, $f_\beta$ is a gap function for $QVI(N^a_\theta, X)$ on $FP(X)$. Now the proof is complete since, for any $x \in FP(X)$, $f_\beta(x) \geq 0$ and $f_\beta(x)$ is null if and only if $x$ is a solution of $QVI(N^a_\theta, X)$ which is equivalent, according to Theorem 4.2, to the fact that $x$ is solution of the generalized Nash equilibrium problem. □

*Remark 4.1* A classic way to reformulate GNEP is to use the so-called regularized Nikaido–Isoda function (see e.g. [9, 13]) defined, for any $\alpha > 0$, by

$$\Psi_\alpha(x,y) := \sum_{\nu=1}^{p} \left[ \theta_\nu\left(y^\nu, x^{-\nu}\right) - \theta_\nu\left(x^\nu, x^{-\nu}\right) + \frac{\alpha}{2}\left\|x^\nu - y^\nu\right\|^2 \right]$$

and, for $x \in \mathbb{R}^n$,

$$V_\alpha(x) := - \inf_{y \in S(x)} \Psi_\alpha(x,y).$$

If $\theta_\nu$ is not convex, but only semistrictly quasiconvex with respect to the $\nu$th variable (as in the above Proposition 4.1), then the regularized Nikaido–Isoda function is of no use. Indeed, for example, properties $(b)$ and $(c)$ of [13, Theorem 2.2] are not satisfied in general and therefore, $V_\alpha$ is no longer a gap function in the more general setting of Proposition 4.1.

The following simple situation provides such an example. We consider a two players generalized Nash game, which is defined by $n_1 = n_2 = 1$ with $\theta_1(x_1, x_2) = x_1|x_1|$, $\theta_2 = 0$ and $X_1(x_2) = [-1; 1]$, $X_2(x_1) = \mathbb{R}$. Then $V_\alpha(0,0) = 0$ but $\theta_1(-1,0) = -1 < \theta_1(0,0)$ which proves that $(0,0)$ is not a solution of the Nash game.

## 5 Conclusions

Except the fact that this study extends to quasivariational inequalities three different concepts of gap functions, another novelty in this paper is that a gap function is proposed for the Generalized Nash Equilibrium problem (GNEP). Additionally, this is done for the semistrictly quasiconvex GNEP case. Error bounds are also provided, under some strong monotonicity assumptions. A natural extension of this work would be to be able to provide analogous error bounds under weaker monotonicity assumptions. This would be of particular interest in order to construct some stopping rule criterion for quasiconvex or pseudoconvex optimization. Another possible extension could be to embed, following [14], the study of gap function into a separation scheme, where the gap function becomes a special case of a separation function in the image space and, hence, a "by-product" of the Hahn–Banach Theorem.

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